$$
\begin{aligned}
N(\bar{c}_1, \bar{c}_2\bar{c}_3\cdots\bar{c}_t) &= N - [N(c_1) + N(c_2) + \cdots + N(c_t)] \\
&\quad + [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + \cdots + N(c_2c_3) + \cdots + N(c_{t-1}c_t)] \\
&\quad - [N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + N(c_1c_3c_4) + \cdots \\
&\qquad + N(c_1c_3c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)] + \cdots + (-1)^t N(c_1c_2c_3\cdots c_t) \\
&= S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t
\end{aligned}
$$

$$
E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t
$$

$$
L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t
$$

If $n \in \mathbf{Z}^+$,

$$
\binom{-n}{r} = \binom{n+r-1}{r}
$$

For all $m, n \in \mathbf{Z}^+,\ a \in \mathbf{R}$,

$$
\begin{aligned}
(1+x)^n &= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n \\
\frac{(1-x^{n+1})}{(1-x)} &= 1 + x + x^2 + x^3 + \cdots + x^n \\
\frac{1}{(1-x)} &= 1 + x + x^2 + x^3 \cdots = \sum_{i=0}^{\infty} x^i \\
\frac{1}{(1+x)^n} &= \binom{-n}{0} + \binom{-n}{1} x + \binom{-n}{2} x^2 + \cdots \\
&= \sum_{i=0}^{\infty} \binom{-n}{i} x^i \\
&= 1 + (-1) \binom{n+1-1}{i} x + (-1)^2 \binom{n+2-1}{2} x^2 + \cdots \\
&= \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i \\
\frac{1}{(1-x)^n} &= \binom{-n}{0} + \binom{-n}{1} (-x) + \binom{-n}{2} (-x)^2 + \cdots \\
&= \sum_{i=0}^{\infty} \binom{-n}{i} (-x)^i \\
&= 1 + (-1) \binom{n+1-1}{i} (-x) + (-1)^2 \binom{n+2-1}{2} (-x)^2 + \cdots \\
&= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i
\end{aligned}
$$

$$
v - e + 2 = r, \quad 3r \le 2e, \quad e \le 3v - 6
$$

$$
\sum deg(v) = 2|E|, \quad \sum deg(R_i) = 2|E|
$$

MACM 201 Final Examination, April 2006

R. Pyke

Aids allowed: nonprogrammable calculator and formula sheet (provided).

Marks for each question are indicated by [ ]. Total marks: 100.

Time allowed: 180 minutes. There are **13** questions.

NAME:

STUDENT NUMBER:

(1) [ ] A standard deck of 52 cards contains cards numbered 1 (Ace), 2, ..., 10, J (Jack), Q (Queen), and K (King). There are 4 suits; clubs, spades, hearts, and diamonds. Each numbered card appears once in each suite (so for example, there are 4 Aces; one each in clubs, spades, hearts, and diamonds). Thus, $52 = 13 \times 4$.

(a) Suppose a hand of 10 cards are dealt out (randomly) from the deck. What is the probability that this hand will contain at least 1 card from each suit?

(b) What is the probability that this hand will contain at least one void (for example, no clubs)?

(2) [ ] Ten students take a math test in a certain room. When the test is over the students leave the room for a break and then return to the room to discuss the answers to the test. If there are 14 chairs in this room, in how many ways can the students seat themselves after the break so that no one is in the same chair that they wrote the test in?

(3) [ ] Solve the following recurrence relation <u>using generating functions</u>.

$$a_{n+1} + 3a_n + 1 \;=\; n^3 + 2^n, \quad a_0 = 1,\ a_1 = 2$$

Check your answer using a_2.

(4) [ ] Solve the following recurrence relation without using generating functions.

$$a_{n+2} - 6a_{n+1} + 9a_n = 2 + 2(3^n), \quad a_0 = -2,\ a_1 = 2$$

Check your answer using a_2.

(5) [ ] Recall that d_n is the number of derangements of $1, 2, \ldots, n$, i.e., how many ways the numbers $1, 2, \ldots, n$ can be arranged so that 1 is not in the first position, 2 is not in the second position, ..., n is not in the n^{th} position.

If $n > 2$, find a recurrence relation satisfied by d_n.

(6) [ ] Consider the two pairs of graphs below; G_1, G_2 and H_1, H_2. Is G_1 and G_2 isomorphic? Is H_1 and H_2 isomorphic? If not, explain why not. If so, exhibit an isomorphism $f(v)$ between the two graphs.

(7) [ ] Find an Euler **trail** and a Hamiltonian **cycle** in the graph below.

(8) [ ] (a) Is the graph G_1 an induced subgraph of G? Explain your answer.

(b) Sketch the complement $\bar{G}$ of G. Begin by arranging the vertices of G along a circle, as given below.

(c) Is G bipartite? Explain.

(d) Is G planar? Explain.

(e) Is the graph below planar? Explain.

(9) [ ] Suppose T is a spanning tree of K_n, the complete graph on n vertices. Prove or disprove: If an edge from e from K_n, $e \notin T$, is added to T to produce the graph T', then T' contains a cycle.

(10) [ ] Prove or disprove: If G is a planar graph on $v \geq 4$ vertices, then $\bar{G}$ is not planar.

(11) [ ] List the vertices in the tree below when they are visited in a preorder traversal and in a postorder traversal.

(12) [ ] Find the depth-first spanning tree for the graph below if the order of the vertices is alphabetical.

(13) [ ] Consider the following graph G below;

(a) Use Dikjistra's Algorithm to find the lengths of the shortest paths from vertex a to all other vertices. What is the shortest path from a to d?

(b) Find a minimal spanning tree for G using Kruskal's Algorithm (assume all edges are undirected). Show the steps of the algorithm.

(c) Find a minimal spanning tree for G using Prim's Algorithm (assume all edges are undirected). Show the steps of the Algorithm.