Midterm I – Solutions

Note Title 06/02/2008

$\boxed{P_1}$ $\displaystyle\int \frac{x^2}{e^{3x^3}}\,dx$

Substitution $u = -3x^3 \Rightarrow du = -9x^2\,dx$

$$\Rightarrow \int \frac{x^2}{e^{3x^3}}\,dx = \int -\frac{1}{9}e^u\,du = -\frac{1}{9}\int e^u\,du = -\frac{1}{9}e^u + C$$

$$= -\frac{1}{9}e^{-3x^3} + C$$

$\boxed{P_2}$ (a) Substitution $u = 2x+1 \Rightarrow du = 2\,dx$

$$\int_1^3 \frac{1}{2x+1}\,dx = \int_3^7 \frac{1}{u}\,\frac{1}{2}\,du = \frac{1}{2}\ln|u|\Big|_3^7 = \frac{1}{2}(\ln 7 - \ln 3) \simeq 0.4236$$

(b)

$$\begin{array}{ccccc} 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 \\ x_0 & x_1 & x_2 & x_3 & x_4 \end{array}$$

$\Delta x = \frac{1}{2}$

$$\int_1^3 f(x)\,dx \simeq \Delta x \left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2} f(x_4)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + \frac{1}{2} f(3)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\,\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{2}\,\frac{1}{7}\right] \simeq 0.4274$$

(c)

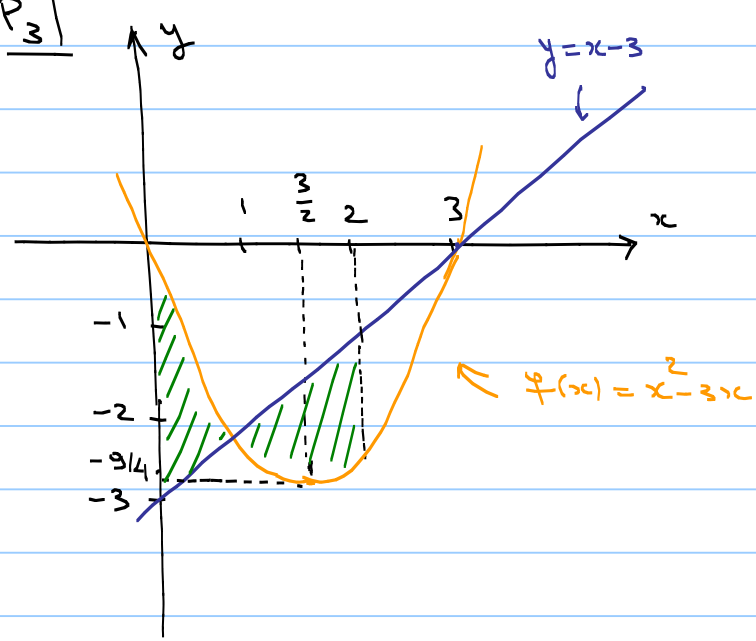
$$\int_1^3 f(x)\,dx \simeq \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{\frac{1}{2}}{3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + f(3)\right]$$

$$= \frac{1}{6}\left[\frac{1}{3} + 4\cdot\frac{1}{4} + 2\cdot\frac{1}{5} + 4\cdot\frac{1}{6} + \frac{1}{7}\right] \simeq 0.4238$$

(d) Simpson's rule give a better approx. (as usual).

$\boxed{P_3}$



the parabola $f(x) = x^2 - 3x$ has a min at $f'(x) = 0$, that is $2x - 3 = 0 \Rightarrow x = \frac{3}{2}$

the value of the min is

$$f\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}\right)^2 - 3\cdot\tfrac{3}{2} = -\left(\tfrac{3}{2}\right)^2 = -\tfrac{9}{4}$$

the function $f(x) = x - 3$ evaluated at $x = \frac{3}{2}$ is $-\frac{3}{2} > -\frac{9}{4}$

These remarks justify the graph above. (We did not give full credit to graphs without justification).

Compute the intersection of the graphs: $x^2-3x = x-3 \Rightarrow$

$$\Rightarrow \quad x^2-4x+3=0 \quad \rightarrow \quad \begin{matrix} x=1 \\ x=3 \end{matrix}$$

The region whose area we need to compute is shaded in green.

In $[0,1]$, $x^2-3x \geq x-3$, while in $[1,2]$, $x-3 \geq x^2-3x$

Therefore,

We write the area as: $\displaystyle\int_0^1 \left[x^2-3x-(x-3)\right]dx + \int_1^2 \left[x-3-(x^2-3x)\right]dx.$

Calculation of the 1st integral:

$$\int_0^1 (x^2-4x+3)\,dx = \left(\frac{x^3}{3} - 4\frac{x^2}{2} + 3x\right)\Big|_0^1 = \frac{1}{3} - 2 + 3 = \frac{4}{3}$$

Calculation of the 2nd integral:

$$\int_1^2 (4x - x^2 - 3)\,dx = \left(4\frac{x^2}{2} - \frac{x^3}{3} - 3x\right)\Big|_1^2 = 2(4-1) - \frac{1}{3}(8-1) - 3$$

$$= 6 - \frac{7}{3} - 3 = \frac{2}{3}$$

Final result: $\frac{4}{3} + \frac{2}{3} = \boxed{2}$

$\boxed{P_4}$ (a) Given a continuous function $f: [a,b] \longrightarrow \mathbb{R}$,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$, for <u>any</u> antiderivative $F(x)$ of f

(b) $$\int_1^5 \frac{\ln\sqrt{x}}{2x}\,dx$$ Make $u = \ln\sqrt{x} \Rightarrow du = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$

$x = 1 \Rightarrow u = \ln 1 = 0$, $x = 5 \Rightarrow u = \ln\sqrt{5}$

$$\Rightarrow \quad du = \frac{1}{2x}\,dx$$

$$\int_1^5 \frac{\ln\sqrt{x}}{2x}\,dx = \int_0^{\ln\sqrt{5}} u\,du = \frac{u^2}{2}\Big|_0^{\ln\sqrt{5}} = \boxed{\frac{1}{2}\left(\ln\sqrt{5}\right)^2}$$

$\boxed{P_5}$ Substitute $u = x+1 \Rightarrow du = dx$, $x = u-1$.

$$\int x\sqrt{x+1}\,dx = \int (u-1)\,u^{\frac{1}{2}}\,du = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du =$$

$$= \frac{1}{1+\frac{3}{2}}\,u^{1+\frac{3}{2}} - \frac{1}{1+\frac{1}{2}}\,u^{1+\frac{1}{2}} + C = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$
