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Simon Fraser University
Department of Mathematics
Final Examination
MATH 157
12 April 2006 8:30am – 11:30am

- Please ensure that you sign your exam above to certify your identity. Unsigned exams will not be marked.
- Only scientific, non-programmable calculators with no differentiation nor integration nor graphing capability are allowed in exams.
- The duration of this exam is 3 hours.
- DO NOT OPEN this test booklet until told to do so.
- Please check that you have all 10 pages of the exam.
- Do ALL your work in this test booklet.
- The value of each question is shown on the left margins.

Question	Score	Maximum
1		6
2		6
3		4
4		6
5		8
6		10
7		8
8		6
9		6
10		6
Total		66

1. Find the following limits.

[2] (a) $\lim_{x\to\infty} \frac{2x+1}{x-1}\left(\frac{x^2}{3x^2-1} - \frac{2(x^3+1)}{x^4+1}\right)$

$$\lim_{x\to\infty} \frac{2x+1}{x-1}\left(\frac{x^2}{3x^2-1} - \frac{2(x^3+1)}{x^4+1}\right) = \frac{2}{1}\cdot\left(\frac{1}{3} - 0\right) = \frac{2}{3}$$

[2] (b) $\lim_{x\to\frac{1}{2}} \frac{2x-1}{\sqrt{2x}-1}$

$$\begin{aligned}
\lim_{x\to\frac{1}{2}} \frac{2x-1}{\sqrt{2x}-1} &= \lim_{x\to\frac{1}{2}} \frac{2x-1}{\sqrt{2x}-1}\,\frac{\sqrt{2x}+1}{\sqrt{2x}+1} \\
&= \lim_{x\to\frac{1}{2}} \frac{(2x-1)(\sqrt{2x}+1)}{(2x-1)} \\
&= \lim_{x\to\frac{1}{2}} \left(\sqrt{2x}+1\right) = 2
\end{aligned}$$

[2] (c) $\lim_{x\to 0^-} \left(e^{1/x} + \frac{x}{\sin x}\right)$

$$\begin{aligned}
\lim_{x\to 0^-} \left(e^{1/x} + \frac{x}{\sin x}\right) &= e^{\lim_{x\to 0^-} \frac{1}{x}} + \lim_{x\to 0^-} \frac{x}{\sin x} \\
&= 0 + 1 = 1
\end{aligned}$$

as $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

2. Find the following derivatives. Do not simplify your answer.

[2] (a) $\frac{dy}{dx}$ where $y = \log_2(2^x + x^2)$

$$\frac{dy}{dx} = \frac{2^x \ln 2 + 2x}{(2^x + x^2)\ln 2}$$

[2] (b) $f'(x)$ where $f(x) = \cos(\sin(1/x))$

$$f'(x) = -\sin(\sin(\frac{1}{x})) \cdot \cos(\frac{1}{x}) \cdot -\frac{1}{x}$$

[2] (c) $f^{(3)}(x)$ where $f(x) = e^{2x}$

$$f^{(1)}(x) = 2e^{2x}, f^{(2)}(x) = 4e^{2x}, f^{(3)}(x) = 8e^{2x}$$

[2] **3.** (a) Write down the explicit conditions you need to check in order for $f(x) = \cos(x^2)$ to be continuous at $x = 3$. Do not verify these conditions.

1. $\cos(x^2)$ is defined at $x = 3$
2. $\lim_{x\to 3} \cos(x^2)$ exists
3. $\lim_{x\to 3} \cos(x^2) = f(3)$

[2] (b) Write down the explicit limit which gives $f'(3)$ for $f(x) = \cos(2x)$. Do not compute the limit.

$$f'(3) = \lim_{x\to 3} \frac{\cos(2x) - \cos(2\cdot 3)}{x-3}$$

4. Consider the following function.

$$f(x) = \begin{cases} x+1+c & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}.$$

[2] (a) Find the value of c so that f is continuous at $x = 0$.

We require that $\lim_{x\to 0^-} x+1+c = c+1 = \lim_x \to 0^+ x^2 = 0 = f(0)$ so $c = -1$.

[4] (b) Is f differentiable at $x = 0$?

No, as

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x-0} = \lim_{x\to 0^-} \frac{(x+1+c) - (1+c)}{x} = 1$$

yet

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2-0}{x-0} = 0.$$

5. Suppose $f(x) = \frac{x^2}{x-1}$, $f'(x) = \frac{x(x-2)}{(x-1)^2}$, $f''(x) = \frac{2}{(x-1)^3}$.

[2] (a) Where is $f(x)$ positive?

$f(x) > 0$ if and only if $x > 1$.

[2] (b) Complete the following table to give the intervals of increase and decrease of $f(x)$.

	x	$(x-2)$	$(x-1)^2$	$f'(x)$	$f(x)$ inc/dec
$(-\infty, 0)$					
$(0, 1)$					
$(1, 2)$					
$(2, \infty)$					

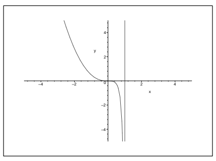
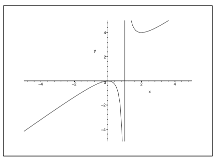
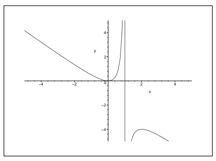
	x	$(x-2)$	$(x-1)^2$	$f'(x)$	$f(x)$ inc/dec
$(-\infty, 0)$	$-$	$-$	$+$	$+$	inc
$(0, 1)$	$+$	$-$	$+$	$-$	dec
$(1, 2)$	$+$	$-$	$+$	$-$	dec
$(2, \infty)$	$+$	$+$	$+$	$+$	inc

[2] (c) Complete the following table to give the intervals of concave up and concave down of $f(x)$.

	$(x-1)^3$	$f''(x)$	$f(x)$ cc up/down
$(-\infty, 1)$			
$(1, \infty)$			

	$(x-1)^3$	$f''(x)$	$f(x)$ cc up/down
$(-\infty, 1)$	$-$	$-$	down
$(1, \infty)$	$+$	$+$	up

[2] (d) Which of the following best represents the graph of $y = f(x)$?

The middle figure best represents the graph of $y = f(x)$.

6. The cost of manufacturing q units of a certain product is given by $C(q) = 100 + 5q - 0.1q^3$ and demand equation is given by $p = 3 - q$ where p is the price per unit and q is the number of units demanded.

[2] (a) Determine the marginal cost function.

$C'(q) = 5 - 0.3q^2$.

[2] (b) Determine the average cost function $\overline{C}(q)$.

$\overline{C}(q) = C(q)/q = (100 + 5q - 0.1q^3)/q$

[2] (c) Determine the revenue function $R(q)$.

$R(q) = pq = (3 - q)q$

[2] (d) Determine the profit function $P(q)$.

$P(q) = R(q) - C(q) = 3q - q^2 - (100 + 5q - 0.1q^3) = -100 - 2q - q^2 + 0.1q^3$

[2] (e) Determine the elasticity of demand as a function of p.

$E = -\frac{p}{q}\frac{dq}{dp} = -\frac{p}{3-p} \cdot -1 = \frac{p}{3-p}$

7. Let $y = f(x) = x^3 + x + 1$.

[2] (a) Determine the Newton iteration formula which solves the equation $f(x) = 0$. Do not attempt to solve the equation.

$f'(x) = 3x^2 + 1$ so the formula is

$$c_{n+1} = c_n - \frac{c_n^3 + c_n + 1}{3c_n^2 + 1}.$$

[2] (b) Use implicit differentiation to determine a formula for $\frac{dx}{dy}$ in terms of x.

$1 = 3x^2 \frac{dx}{dy} + \frac{dx}{dy}$ so $\frac{dx}{dy} = \frac{1}{3x^2+1}$

[2] (c) Suppose x and y as related above are functions of t. Find $\frac{dx}{dt}$ in terms of $x, \frac{dy}{dt}$.

$\frac{dy}{dt} = 3x^2 \frac{dx}{dt} + \frac{dx}{dt}$ so $\frac{dx}{dt} = \frac{\frac{dy}{dt}}{3x^2+1}$

[2] (d) Give an approximation to $f(x)$ at $x = 1 - 0.1$ using differentials.

Let $x = 1, \Delta x = dx = -0.1$. Then $\Delta y \approx dy = f'(x)dx = (3x^2 + 1)dx = 4 \cdot -0.1 = -0.4$ so $f(x + \Delta x) \approx f(x) - 0.4 = 3 - 0.4 = 2.6$.

8. Suppose the revenue R from the sales of q units of a certain product is given

$$R(q) = -q^2(q-3) + 10.$$

[4] (a) Calculate the absolute maximum of $R(q)$ on $[0, 5]$.

$R(q) = -q^2(q-3) + 10 = -q^3 + 3q^2 + 10$ so $R'(q) = -3q(q-2)$. The critical numbers in $[0, 5]$ are $0, 2$. We have that $f(0) = 10, f(2) = 14, f(5) = -40$ so 14 is an absolute maximum for f on $[0, 5]$.

[2] (b) Calculate the value of q such that $R(q)$ has a point of diminishing return at q.

$R'(q) = -3q^2 + 6q$ so $R''(q) = -6q + 6$. R has an inflection point at $q = 1$ so R has a point of diminishing return here.

9. Your client has \$1,000 on 1 July 2005 in an account at bank A which earns 2.0% compounded continuously. Bank B wishes to entice your client by offering \$1 to switch to their account which earns 2.0% compounded semi-annually.

[2] (a) Calculate the value of your client's account after a time period of t years from 1 July 2005 if the \$1,000 is left in bank A.

$A_1 = 1000e^{0.02t}$

[2] (b) Calculate the value of your client's account after a time period of t years from 1 July 2005 if the \$1,000 is transferred to bank B and assuming the bonus of \$1 is deposited in the new account at the time of transfer.

$A_2 = (1000 + 1)(1 + 0.02/2)^{2t}$

[2] (c) Which account will offer your client the larger future value in the long term?

$1000e^{0.2t} > 1001(1.01)^{2t}$ if and only if $0.2t + \ln(1000) > 2t\ln(1.01) + \ln(1001)$ if and only if $0.18t > \ln(1001) - \ln(1000)$. This holds if $t > 0.005$. So after 0.005 years, the account in bank A has the greater value.

10. Consider a loan of $250,000 amortized in two different ways:

(A) over a period of 25 years at an interest rate of 6.0% compounded monthly

(B) over a period of 20 years at an interest rate of 7.0% compounded monthly

Put the answer to the following questions in the boxes where appropriate. Only your final answer will be graded. Please round your answer to the nearest cent.

[2] (a) Find the monthly payment for each loan.

Loan (A): $n = 25 \cdot 12, i = 0.06/12$ so $R = 250000/((1-(1+i)^{-n})/i) \approx 1610.75$.
Loan (B): $n = 20 \cdot 12, i = 0.07/12$ so $R = 250000/((1-(1+i)^{-n})/i) \approx 1938.25$.

Loan (A):
Loan (B):

[2] (b) Find the total interest charges for each loan.

Loan (A): $1610.75 \cdot 25 \cdot 12 - 250000 \approx 233226.05$
Loan (B): $1938.25 \cdot 20 \cdot 12 - 250000 \approx 215179.37$

Loan (A):
Loan (B):

[2] (c) Find the amount owed at the end of 5 years for loan (A).

This is given by the present value of a 20 year annuity. $P = R(1-(1+i)^{-20\cdot 12})/i \approx 224830.22$