SIMON FRASER UNIVERSITY

# MATH 157 Final Exam

**D100 & D300**

**December 12, 2007**

**Time: 08:30 – 11:00 ($2\frac{1}{2}$ hours)**

Last Name SOLUTION

Given Name(s) ______________________

Student # ______________________

SFU email ID ______________________

*Good Luck!*

Student signature

INSTRUCTIONS

1. **Do not open this booklet until told to do so!**
2. This exam has 10 questions on 12 pages excluding this cover page. Please check to make sure your exam is complete.
3. **Calculators are not allowed. No electronic devices may be within reach of a student.**
4. Write your full name, student number and SFU email ID on the cover page.
5. Please read the questions carefully, and make sure you understand what you are asked to do!
6. Please write with a black or blue **pen**.
7. Write your answers in the space provided. Use the reverse side of the **previous page** if you need more room for your rough work.
8. **You may lose marks if your explanations are incomplete, missing, or poorly presented.**
9. **You must stop writing immediately when asked to do so!**

| Question | Score |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| 6 | /10 |
| 7 | /10 |
| 8 | /10 |
| 9 | /10 |
| 10 | /10 |
| Total | /100 |

**1. Agony of Choice.** The following questions are multiple choice. No explanation is required for your answer.

[2] (a) $f(x) = xe^{-x^2}$. Then $f'(x) =$

A: $\square$ $e^{-x^2}$, B: $\square$ $(1-2x^2)e^{-x^2}$,

C: $\square$ $e^{-x^2} - 2xe^{-x^2-1}$, D: $\square$ $e^{-x^2} - 2xe^{-x^2}$,

E: $\square$ None of the above

[2] (b) $f^{(4)}(x) = 0$ for all $x$, i.e., the fourth derivative of $f$ is zero everywhere. Then

A: $\square$ $f(x) = 0$, for all $x$, B: $\square$ $f$ has infinitely many critical points,

C: $\square$ $\lim_{x\to\infty} f(x) = 0$, D: $\square$ $f$ is a polynomial of degree $\leq 3$,

E: $\square$ None of the above

[2] (c) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} =$

A: $\square$ does not exist, B: $\square$ $0$, C: $\square$ $1$, D: $\square$ $\frac{1}{2}$,

E: $\square$ None of the above

[2] (d) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x+3} =$

A: $\square$ does not exist, B: $\square$ $\infty$, C: $\square$ $6$, D: $\square$ $\frac{1}{3}$,

E: $\square$ None of the above

[2] (e) $f(x) = \ln\sqrt{1+x^2}$. Then $f'(x) =$

A: $\square$ $\dfrac{1}{\sqrt{1+x^2}}$, B: $\square$ $\dfrac{x}{1+x^2}$, C: $\square$ $\dfrac{2x}{\sqrt{1+x^2}}$, D: $\square$ $\dfrac{1}{2(1+x^2)}$,

E: $\square$ None of the above

## 2. Implicit and explicit.

[7] (a) Let $f(x) = (1+x)^{\ln x}$, for $x > 0.$. Taking the logarithm on both sides of the equation gives an **implicit** definition of $f$, namely

$$\ln f(x) = (\ln x)(\ln(1+x)),$$

again for $x > 0$. Now use implicit differentiation. Choose one of the answers below, but you **must show your work** to receive any credit. $f'(x) =$

A: ☐ $\frac{\ln x}{x}(1+x)^{(\ln x)-1}$, B: ☑ $\left(\frac{1}{x}\ln(1+x) + \frac{1}{1+x}\ln x\right)(1+x)^{\ln x}$,

C: ☐ $\left(\frac{1}{x+1}\ln(1+x) + \frac{1}{x}\ln x\right)(1+x)^{\ln x}$, D: ☐ $\ln x(1+x)^{(\ln x)-1}$,

E: ☐ None of the above

$$\ln f(x) = \ln x \ln(1+x)$$

$$\frac{d}{dx} \quad \frac{f'(x)}{f(x)} = \frac{1}{x}\ln(1+x) + \frac{1}{1+x}\ln x$$

$$f'(x) = \left\{\frac{1}{x}\ln(1+x) + \frac{1}{1+x}\ln x\right\}\underbrace{f(x)}_{=(1+x)^{\ln x}}$$

[3] (b) The population of a small city $t$ years from now is predicted to be

$$N(t) = 20000 + \frac{10000}{(t+2)^2}.$$

Find the population in the long run, that is $\lim_{t\to\infty} N(t)$.

$$\lim_{t\to\infty}\left\{20000 + \frac{10000}{\underbrace{(t+2)^2}_{\to\infty}}\right\} = 20000$$

SHOW YOUR WORK

**3. Vintage Calculus - Production Cost.** A manufacturer of handcrafted wine racks has determined that the total cost to produce $x$ units per month is given by

$$C(x) = 100 + 15x - x^2, \ (2 \leq x \leq 8).$$

$\overline{C}(x) = \frac{100}{x} + 15 - x$

The **total cost per unit** is hence $\overline{C}(x) = C(x)/x$. Make sure you understand which question is about $C(x)$ and which is about $\overline{C}(x)$. One unit is a box of 24 racks, so it makes sense to produce fractional units, for example 7.5 units. Minimum production is 2 units, and the maximum production capacity is 8 units. The following questions are multiple choice. No explanation is required for your answer.

$C'(x) = 15 - 2x$ $\qquad$ $\overline{C}'(x) = -\frac{100}{x^2} - 1 < 0$

[1] (a) The average rate of change of **total cost per unit** for manufacturing between 4 and 6 units is

A: ☐ $\frac{31}{3}$, B: ☐ $\frac{31}{6}$, C: ☑ $-\frac{31}{6}$, D: ☐ $-\frac{31}{3}$, E: ☐ Other

[1] (b) The instantaneous rate of change of **total cost per unit** with respect to the number of units produced when 5 units are produced is

A: ☑ $-5$, B: ☐ $5$, C: ☐ $4$, D: ☐ $-\frac{31}{6}$, E: ☐ Other

[1] (c) The instantaneous rate of change of **total cost** with respect to the number of units produced when 5 units are produced is

A: ☐ $-5$, B: ☑ $5$, C: ☐ $4$, D: ☐ $\frac{31}{6}$, E: ☐ Other

[1] (d) The additional cost if production is increased from 5 units to 6 units is

A: ☐ $-5$, B: ☐ $5$, C: ☑ $4$, D: ☐ $\frac{31}{6}$, E: ☐ Other

[1] (e) The answer in part (c) is the linear approximation to the answer in part (d).

A: ☑ **True**, B: ☐ **False**

$C'(x) = 0$

$15 - 2x = 0$

$x = \frac{15}{2}$

BUT $C''(x) = -2 < 0$

[1] (f) The answer in part (c) is also known as the "marginal cost" at $x = 5$.

A: ☑ **True**, B: ☐ **False**

→ MAXIMUM

END PTS $C(2) = 126$

$C(8) = 156$

[2] (g) At what level should production be fixed to minimize **total cost**?

A: ☐ 8, B: ☐ $\frac{15}{2}$, C: ☐ 4, D: ☑ 2, E: ☐ Other

$\overline{C}'(x) < 0$

→ DECREASING

[2] (h) Which production level minimizes **total cost per unit**?

A: ☑ 8, B: ☐ $\frac{15}{2}$, C: ☐ 4, D: ☐ 2, E: ☐ Other

**4. No test without bicycles**. [Please note this problem is on two pages!] First there was the Exhilarator, then the Hummerator. In search of new markets our bike company is poised to start production on a sleek racing bike, called the "Racer 157". Management estimates the monthly demand function for the "Racer 157" as

$$p = 900 - 2x,$$

where $x$ is the number of bicycles produced and sold per month, and price $p$ is in Dollars. The monthly cost function to produce the "Racer 157" is linear, with a monthly fixed cost of \$6000, and a marginal cost of \$500. The company has a **maximum monthly production capacity of 120** "Racer 157" bicycles.
The company anticipates a government subsidy being introduced in the near future. Assume this subsidy to be $s$ Dollars per bicycle.

$x \leq 120$

[1] (a) What is $R(x)$, the monthly revenue as a function of $x$?

$$R(x) = x\left(900 - 2x\right)$$

[1] (b) What is $C(x)$, the monthly cost as a function of $x$? Include the subsidy $s$ in the cost!

$$C(x) = 6000 + (500 - s)x$$

[1] (c) What is $P(x)$, the monthly **PROFIT** as a function of $x$? Include the subsidy!

$$P(x) = -2x^2 + (400 + s)x - 6000$$

[3] (d) Determine the level of output and price at which profit is maximized. Your answers should depend on $s$.

$$P'(x) = -4x + 400 + s \longrightarrow x = 100 + \frac{s}{4}$$

FOR $s > 80$, $x = 100 + \frac{s}{4} > 120$, OUTSIDE INTERVAL

$s > 80$ MAXIMIZED AT $x = 120$, $p = 660$

$s \leq 80$ MAXIMIZED AT $x = 100 + \frac{s}{4}$, $p = 700 - \frac{s}{2}$

*This question continues on page 5*

*Continuation of problem 4.*

[1] (e) Without subsidy, i.e. $s = 0$, what is the level of output and price at which profit is maximized?

$x = 100$ , $p = 700$

[1] (f) A few months into the production run, the government, in its support for fitness and non-polluting transportation decides to provide the company with a $100 subsidy per bicycle ($s = 100$). what is the new level of output and price at which profit is maximized?

$S = 100 > 80 \rightarrow \begin{cases} x = 120 \\ p = 660 \end{cases}$

[1] (g) How much of this $100 per bicycle subsidy effectively goes to the company, how much to the consumer?

$60 → COMPANY , $40 → CONSUMER

[1] (h) You are advising the government on the best level for the subsidy. Goals are increased production and a better price for the consumer, but the government does not want to spend money on the subsidy which will only go to the company. What is your recommendation to the government?

S= $80 ; higher subsidy does not affect price

SHOW YOUR WORK

5. **Elastic and profitable books** After the unexpected fabulous success of "*How to Pass Math157 Without Paying Attention in Class*" the business students' most favourite authors MC & MT wrote another award winning paperback entitled "*Top Ten Things Every Math 157 Student Should Know Before Writing the Final Exam*". Unfortunately, due to unforeseeable customs delays the book is not expected to hit the shelves until after December 12, yet still in time for the lucrative Christmas shopping season. You guessed it - you are lucky enough to have a preprint of the booklet in front of you!
The monthly demand function of the booklet is given by $1000p + q = 6000$ where $q$ is the quantity and $p$ is the price in Dollars.
Monthly fixed cost for printing and distributing the booklet are \$1000, and the marginal cost of printing and distributing one booklet is \$1.

[3] (a) Find the elasticity of demand, $E$. $q = 6000 - 1000p = 1000(6-p)$

$$\frac{dq}{dp} = -1000, \quad E = -\frac{p}{q}\frac{dq}{dp} = -\frac{p}{1000(6-p)}(-1000) = \underline{\underline{\frac{p}{6-p}}}$$

[2] (b) At what price will the demand have unit elasticity, i.e., $E = 1$?

$$E = \frac{p}{6-p} = 1 \qquad p = 6-p \qquad \boxed{p=3}$$

[1] (c) At what price will the authors realize maximum revenue?

WHEN $E=1$, $\boxed{p=3}$

[4] (d) Which quantity $q$ and price $p$ maximize profit, and what is the maximum profit?

Profit $W(p) = pq - q - 1000 = q(p-1) - 1000$

$W(p) = 1000\left[(6-p)(p-1) - 1\right] = 1000\left[-p^2 + 7p - 7\right]$

$W'(p) = 1000\left[-2p + 7\right]$, $W'(p) = 0 \to p = 3.5$

$W''(p) = -2000 < 0 \to$ MAX $\qquad q = 250$

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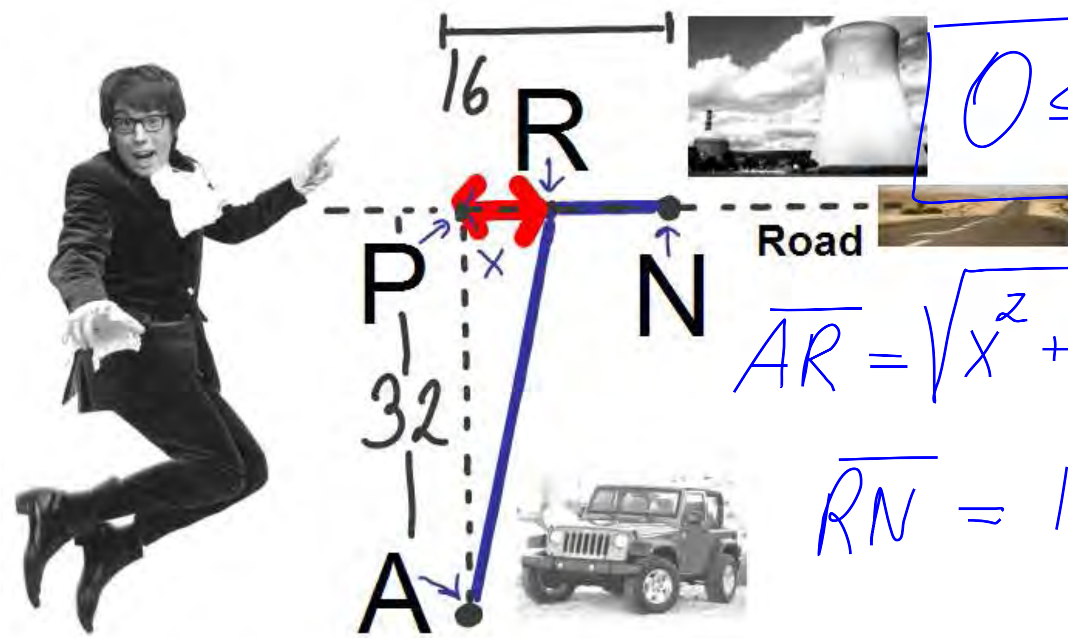
SHOW YOUR WORK $W(3.5) = 1000\left[\underbrace{\underbrace{(6-3.5)}_{2.5}\underbrace{(3.5-1)}_{2.5}}_{6.25} - 1\right]$

PRICE $p = \$3.50$

QUANT $q = 2500$

PROFIT $= \underline{\underline{\$5250}}$

**6. Groovy, Baby – Showdown at High Noon**. [Please note this problem is on two pages!] It is 11:10 am. Austin Powers is in a jeep, driving through the sandy desert, checking his almost perfect teeth in the rear mirror - rrrrrrrhrr! He is at point A, 32 km from the nearest straight paved road which runs east-west; the nearest point on the road, point P is 32 km north of A. Down the road (16km east of point P, see drawing) is a nuclear power plant (point N), where Dr Evil has set a time bomb to explode exactly at noon. The jeep can travel 48 km/h in the sand, and 80 km/h on the paved road. Austin must choose the best possible route, so that there is still time left to defuse the bomb. He does one of his famous shaggadelic calculations, but gets lost in a mirage of psychedelic images from the 60s and distorted pictures of the equation $\sqrt{5} \approx 2.236$. You - you tell Austin exactly which route to take, and how much time he will have left to deal with the bomb. And – be behaved....




$\boxed{0 \le x \le 16}$

$\overline{AR} = \sqrt{x^2 + 32^2}$

$\overline{RN} = 16 - x$

[1] (a) If Austin drove 32 km the shortest way across the sand to the paved road (A to P), and then 16 km down the paved road (P to N), then when would he arrive at nuclear power plant?

$\frac{32}{48} + \frac{16}{80}$ hours

$= \left(\frac{2}{3} + \frac{1}{5}\right)$ hours $= 40 + 12$ min $= 52$ minutes

$\rightarrow$ arrives at 12:02 pm, TOO LATE

*This question continues on page 8*

*Continuation of problem 6.*

[2] (b) Let $x = \overline{PR}$ be the distance from point P to point R. As a function of $x$, what is the time $t(x)$ required to drive straight from point A to point R, turning right, and then drive straight from point R to N (see map)? Naturally, Austin goes full speed!

$$t(x) = \frac{1}{48}\left(x^2+32^2\right)^{1/2} + \frac{1}{80}(16-x) \quad \text{hours}$$

[5] (c) Minimize driving time $t(x)$.

$$t'(x) = \frac{1}{48}\cdot\frac{1}{2}\left(x^2+32^2\right)^{-1/2}\cdot 2x - \frac{1}{80} = \frac{1}{48}\frac{x}{\sqrt{x^2+32^2}} - \frac{1}{80}$$

$$t'=0: \quad \frac{1}{3}\frac{x}{\sqrt{x^2+32^2}} = \frac{1}{5} \rightarrow 5x = 3\sqrt{x^2+32^2} \rightarrow 25x^2 = 9x^2 - 9\cdot 32^2$$

$$16x^2 = 3^2\cdot 32^2 \rightarrow 4x = 3\cdot 32 \rightarrow x = 24 \quad \text{OUT OF DOMAIN } 0\le x\le 16$$

END POINTS: $t(16) = \frac{16}{48}\sqrt{5} = \frac{1}{3}\sqrt{5}$ hours $= 20\sqrt{5}$ minutes $\approx 44.72$

In a) $t(0) = 52$ minutes $\rightarrow$ MIN FOR $x = 16$

[2] (d) How much time will Austin have to defuse the bomb? Pick the correct answer for his **arrival time** below - you **must show your work!**

A: ☐ before 11:54, C: ☐ between 11:56 and 11:58, E: ☐ after noon (BOOM!)

B: ☑ between 11:54 and 11:56, D: ☐ between 11:58 and noon

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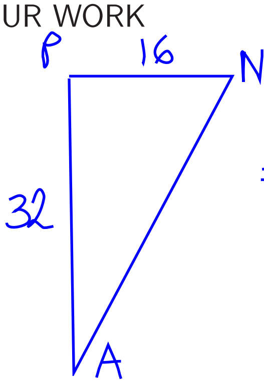
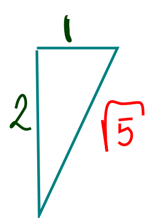
SHOW YOUR WORK




$= 16\times$

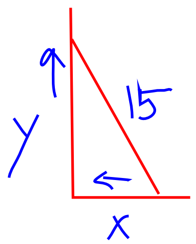



$\longrightarrow \overline{AN} = 16\sqrt{5}$

## 7. Related and not related

[6] (a) A $15\ m$ long ladder is placed against a large building. The bottom of the ladder is placed $10\ m$ away from the wall and is being pushed towards the wall at a rate of $\frac{1}{12}\ m/sec$. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?

$$x = x(t),\ y = y(t)$$




$$x^2 + y^2 = 15^2$$

$$2x x' + 2y y' = 0 \rightarrow \boxed{y' = -\frac{x}{y} x'}$$

After 12 seconds:

$$x = 10 - \left(\tfrac{1}{12}\right)(12) = 9m,\quad y = \sqrt{15^2 - 9^2} = 3\sqrt{5^2 - 3^2} = 12m$$

$$y' = -\frac{x}{y} x' = -\frac{9}{12}\left(-\frac{1}{12}\right) = -\frac{3}{4}\left(-\frac{1}{12}\right) = \frac{1}{16}\ m/sec$$

[4] (b) We have the trigonometric identity $\sin(2x) = 2\cos(x)\sin(x)$.
Using differentiation find a trigonometric identity expressing $\cos(2x)$ in terms of $\cos(x)$ and $\sin(x)$.

$$\sin(2x) = 2\cos x \sin x$$

$$\frac{d}{dx}\qquad 2\cos(2x) = 2(-\sin x)\sin x + 2\cos x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

SHOW YOUR WORK

## 8. Critical

[5] (a) Find all critical values of the function $f(x) = |x^2 - x - 2|$ on the interval $[-5, 5]$.

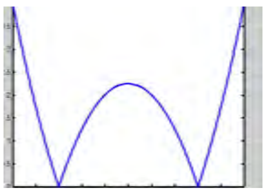
$g(x) = x^2 - x - 2$, $g'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$= (x-2)(x+1)$ $g'(2) = 3 \neq 0$, $g'(-1) = -3 \neq 0$

$\Rightarrow f'$ DOES NOT EXIST AT $x = 2$, $x = -1$

$f'(\frac{1}{2}) = 0$

$-1, \frac{1}{2}, 2$



[5] (b) Find all critical values of the function $f(x) = x^{\frac{2}{3}}(x - 2)$ on the interval $[-5, 5]$.

$$f'(x) = \frac{2}{3}x^{-1/3}(x-2) + x^{2/3}$$

$\rightarrow$ $f'$ does not exist at $\boxed{x=0}$

$$f' = 0: \frac{2}{3}x^{-1/3}\left(x - 2 + \frac{3}{2}x\right) = \frac{2}{3}x^{-1/3}\left(\underbrace{\frac{5}{2}x - 2}_{=0}\right)$$

$\rightarrow$ $x = \frac{4}{5}$ $[0, 4/5]$

SHOW YOUR WORK

## 9. Famous people

[5] (a) **Sir Isaac Newton, 1643-1727**. Let $p(x) = x^3 - 6x^2 + 5x + 12$. Starting with $c_0 = 1$, perform two iterations of Newton's method (i.e., compute $c_1$, and $c_2$). Have you come close to a solution?

$$p'(x) = 3x^2 - 12x + 5$$

$$c_{n+1} = c_n - \frac{p(c_n)}{p'(c_n)}$$

$p(1) = 12$, $p'(1) = -4$

$c_1 = 1 - \frac{12}{-4} = 4$; $p(4)=0 \rightarrow \underline{\underline{c_2 = 4}}$

$c_1$ is a solution

[5] (b) **Guillaume d'Hôpital** (although it probably was **Johann Bernoulli**'s idea...). Consider the function

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$

Explain why $f$ is continuous at $x = 0$. Does the derivative of $f$ exist at $x = 0$? If yes, then compute its value using the definition of the derivative as a limit, if not explain why not!

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{\sin x}{1} = 0$$

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SHOW YOUR WORK

so $\lim_{x \to 0} f(x) = f(0)$ : Continuous $\underline{\text{at } x=0}$

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{1-\cos h}{h^2} =$$

$$= \lim_{h \to 0} \frac{\sin h}{2h} = \lim_{h \to 0} \frac{\cos h}{2} = \frac{1}{2}$$

$$\longrightarrow f'(0) = \frac{1}{2}$$

**10. A little bit of everything** Let the function $f(x) = x^3 - 9x$, be defined on the interval $I = [-3, 5]$.

[2] (a) Find all zeros of $f$.

$f = x(x^2-9) = x(x-3)(x+3)$

$-3, 0, 3$

[2] (b) Find all critical points of $f$ on $I$.

$f' = 3x^2 - 9$

$f' = 0: \quad x = \pm\sqrt{3}$

$f''(x) = 6x$

$< 0$ for $x < 0$

$> 0$ for $x > 0$

[1] (c) Find all local minimizers of $f$ on $I$.

$x = +\sqrt{3}$

[1] (d) Find all local maximizers of $f$ on $I$.

$x = -\sqrt{3}$

| $x$ | $-3$ | $-\sqrt{3}$ | $\sqrt{3}$ | $5$ |
|---|---|---|---|---|
| $f(x)$ | $0$ | $12\sqrt{3}$ | $-12\sqrt{3}$ | $80$ |

[1] (e) Does $f$ have an absolute minimum on $I$? If yes, what is that minimum, if no, explain why not.

Yes, at $+\sqrt{3}$, value $= -12\sqrt{3}$

[1] (f) Does $f$ have an absolute maximum on $I$? If yes, what is that maximum, if no, explain why not.

yes, at end point 5, value = 80

[1] (g) Does $f$ have a point of inflection in $I$? If yes, for which value of $x$, if no, explain why not.

$f''(0) = 0$, $f''$ changes sign at 0: $\boxed{x=0}$

[1] (h) What is the largest open interval <u>contained in $I$</u> on which $f$ is concave upward?

The interval $(0, 5)$

SHOW YOUR WORK

# Math 157 Final Exam Work Sheet

Note Title 12/12/2007