Simon Fraser University
Department of Mathematics
Burnaby Campus

MATH 157-3, 1107
Final Examination
December 9th, 2010, 8:30 – 11:30

PROVIDE THIS DATA AS IT APPEARS ON **WebCT**!

Last Name (please print): ____________**KEY**_________________

First Name (please print): ____________________________________

SFU Student Number: ____________________________________

SFU email ID: _________________________@sfu.ca

Instructor: P. Menz

Instructions:

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.
2. Fill in the above box.
3. This exam contains 14 pages with a total of 10 questions. Once the exam begins please check to make sure your exam is complete.
4. SHOW ALL YOUR WORK!
5. If you run out of space in a problem, use the space on the back of the cover page and clearly indicate where the solution continues.
6. **Only** scientific, non-programmable calculators with no differentiation and integration capabilities are allowed.
7. No book, paper, or device, other than the usual writing instruments, this booklet and an acceptable calculator, shall be within reach of a student during the examination.
8. During the examination, speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.

Do not write in this table!

Question	**Marks**
1	**/10**
2	**/12**
3	+ **/20**
4	**/6**
5	**/5**
6	**/7**
7	**/9**
8 a-c	**/9**
8 d	**/4**
9	**/8**
10	**/10**
Total	**/100**

(left blank intentionally)

1. Answer **T** (true) or **F** (false) in the boxes provided. No explanation is necessary. **[1 mark each = 10 marks]**

a) F Given any two points (x_1, y_1) and (x_2, y_2) on a line, then the slope of the line is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

b) F If $f(a)$ is defined and $\lim\limits_{x \to a} f(x)$ exists, then f is continuous at $x = a$.

c) T $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^-} f(x) = L = \lim\limits_{x \to a^+} f(x)$, where L is a real number.

d) F The instantaneous rate of change of a function $y = f(x)$ at x is defined as $\lim\limits_{x \to h} \dfrac{f(x-h) + f(x)}{h}$ provided this limit exists.

e) F $\lim\limits_{n \to \infty} (1+n)^{1/n} = e$.

f) T An absolute extremum of a continuous function $y = f(x)$ on $[a,b]$ must occur at either the endpoints of the domain or a critical number $c \in [a,b]$ of f.

g) F If the demand is inelastic then total revenue increases as price decreases.

h) F The Linear Approximation of a function f near $x = a$ is given by $L(x) = f(a) - f'(a)(x+a)$.

i) T The iterative formula for Newton's Method is given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where x_1 is the initial value, $n \in \mathbb{N}$, and $f'(x_n) \neq 0$.

j) T The Present Value of an Ordinary Decreasing Annuity is given by $P = R \cdot a_{\overline{n}|i} = R\left[\dfrac{1-(1+i)^{-n}}{i}\right]$.

2. Find the following limits if they exist. **[3 marks each = 12 marks]**

a) $$\lim_{x\to 6^-}\frac{x^2-36}{|x-6|}=\lim_{x\to 6^-}\frac{(x-6)(x+6)}{-(x-6)}=\lim_{x\to 6^-}-(x+6)=-12$$

b) $$\lim_{x\to 0}\frac{\sin 8x}{4x}=2\lim_{x\to 0}\frac{\sin 8x}{8x}=2(1)=2$$

c) $$\lim_{x\to 1}\frac{x^7-1}{x^{39}-1}=\lim_{x\to 1}\frac{(x-1)\left(x^6+x^5+...+1\right)}{(x-1)\left(x^{38}+x^{37}+...+1\right)}=\lim_{x\to 1}\frac{x^6+x^5+...+1}{x^{38}+x^{37}+...+1}=\frac{7}{39}$$

d) $$\lim_{x\to -\infty}\frac{1+2x-3x^2+4x^3-5x^4-6x^5}{5+4x-3x^3+2x^5}$$

$$=\lim_{x\to -\infty}\frac{\frac{1}{x^5}+\frac{2}{x^4}-\frac{3}{x^3}+\frac{4}{x^2}-\frac{5}{x}-6}{\frac{5}{x^5}+\frac{4}{x^4}-\frac{3}{x^2}+2}$$

$$=\frac{-6}{2}=-3$$

3. Find the following derivatives. **[4 marks each = 20 marks]**

a) $f(x)=\sin\left(e^{\sqrt{x}}\right)$, $f'(x)$ **Do not simplify!**

$$f'(x)=\cos\left(e^{\sqrt{x}}\right)e^{\sqrt{x}}\frac{1}{2\sqrt{x}}$$

b) $y=x^{100}100^{x}, \dfrac{d^2y}{dx^2}$ **Do not simplify!**

$$y=x^{100}100^{x},$$

$$\frac{dy}{dx}=100x^{99}100^{x}+x^{100}100^{x}\ln 100$$

$$\frac{d^2y}{dx^2}=100\left(99x^{98}100^{x}+x^{99}100^{x}\ln 100\right)+\ln 100\left(100x^{99}100^{x}+x^{100}100^{x}\ln 100\right)$$

c) $g(x)=\dfrac{\ln(\sin x)}{\cos x}, g'\left(\dfrac{\pi}{4}\right)$ **Evaluate exactly!**

$$g'(x)=\frac{\dfrac{(\cos x)^2}{\sin x}-\ln(\sin x)(-\sin x)}{(\cos x)^2}$$

$$g'\left(\frac{\pi}{4}\right)=\frac{\dfrac{\left(\cos\frac{\pi}{4}\right)^2}{\sin\frac{\pi}{4}}-\ln(\sin\frac{\pi}{4})\left(-\sin\frac{\pi}{4}\right)}{\left(\cos\frac{\pi}{4}\right)^2}=\frac{\frac{1}{\sqrt{2}}+\ln(\frac{1}{\sqrt{2}})\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{2}}$$

$$=\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}}\ln(\frac{1}{\sqrt{2}})=\sqrt{2}+\sqrt{2}\ln(\frac{1}{\sqrt{2}})$$

d) $y=\dfrac{\sqrt{7x+1}\left(7x+2\right)^2\left(7x+3\right)^3}{\left(7x+4\right)^4\left(7x+5\right)^5},\ y'(0)$ **Evaluate exactly!**

$$\ln y=\ln\frac{\sqrt{7x+1}\left(7x+2\right)^2\left(7x+3\right)^3}{\left(7x+4\right)^4\left(7x+5\right)^5}$$
$$=\frac{1}{2}\ln\left(7x+1\right)+2\ln\left(7x+2\right)+3\ln\left(7x+3\right)-4\ln\left(7x+4\right)-5\ln\left(7x+5\right)$$

$$\frac{d}{dx}\left(\ln y\right)=\frac{d}{dx}\left(\frac{1}{2}\ln\left(7x+1\right)+2\ln\left(7x+2\right)+3\ln\left(7x+3\right)-4\ln\left(7x+4\right)-5\ln\left(7x+5\right)\right)$$
$$\frac{1}{y}y'=7\left(\frac{1}{2\left(7x+1\right)}+\frac{2}{7x+2}+\frac{3}{7x+3}-\frac{4}{7x+4}-\frac{5}{7x+5}\right)$$
$$y'=7\frac{\sqrt{7x+1}\left(7x+2\right)^2\left(7x+3\right)^3}{\left(7x+4\right)^4\left(7x+5\right)^5}\left[\frac{1}{2\left(7x+1\right)}+\frac{2}{7x+2}+\frac{3}{7x+3}-\frac{4}{7x+4}-\frac{5}{7x+5}\right]$$

$$y'(0)=7\frac{1(2)^2(3)^3}{(4)^4(5)^5}\left[\frac{1}{2}+\frac{2}{2}+\frac{3}{3}-\frac{4}{4}-\frac{5}{5}\right]=\frac{7\cdot 2\cdot 27}{800000}=\frac{189}{400000}$$

e) $y=x^{\ln x},\dfrac{dy}{dx}$ **Write in terms of *x* only!**

$$\ln y=\ln x^{\ln x}=\ln x\ln x=\left(\ln x\right)^2$$
$$\frac{d}{dx}\left(\ln y\right)=\frac{d}{dx}\left(\left(\ln x\right)^2\right)$$
$$\frac{1}{y}\frac{dy}{dx}=\frac{2\ln x}{x}$$
$$\frac{dy}{dx}=y\frac{2\ln x}{x}=x^{\ln x}\frac{2\ln x}{x}$$

4. Trigonometric Functions: **[2+4 marks= 6 marks]**

a) Evaluate the expression $\csc\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$. (textbook exercise 6.3 #20)

Let $\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, where the angle $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This implies that $\sin\theta = \frac{1}{\sqrt{2}}$ and so $\theta = \frac{\pi}{4}$. Finally, $\csc\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) = \csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$.

b) Find the equation of the tangent line to the graph of the function $f(x) = \tan(x)$ at the point $x = \frac{5}{6}\pi$.

We need a slope and a point.

Point:

$$\left(\frac{5}{6}\pi, f\left(\frac{5}{6}\pi\right)\right) = \left(\frac{5}{6}\pi, \tan\left(\frac{5}{6}\pi\right)\right) = \left(\frac{5}{6}\pi, -\frac{1}{\sqrt{3}}\right)$$

Slope:

$$f'(x) = \sec^2(x)$$

$$f'\left(\frac{5}{6}\pi\right) = \sec^2\left(\frac{5}{6}\pi\right) = \left(-\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

Tangent line equation:

$$y - \left(-\frac{1}{\sqrt{3}}\right) = \frac{4}{3}\left(x - \frac{5}{6}\pi\right) \Rightarrow y + \frac{1}{\sqrt{3}} = \frac{4}{3}x - \frac{10}{9}\pi$$

5. The world population at the beginning of 1990 was 5.3 billion. Assume that the population continues to grow exponentially at its present rate of approximately 2%/year. (textbook exercise 5.5 #12 and #13) **[5 marks]**

   a) Find the function *Q(t)* that expresses the world population (in billions) as a function of time *t* (in years), with $t=0$ corresponding to the beginning of 1990.

   $Q(t)=y_0e^{kt}$

   We know $y_0=5.3$ and $1.02\times 5.3=5.3e^{1k}$.

   Solving for *k* we get

$$1.02=e^{1k}$$
$$k=\ln 1.02$$

   Finally, $Q(t)=5.3e^{t\cdot\ln 1.02}=5.3e^{\ln 1.02^t}=5.3(1.02)^t$.

   b) Find the length of time to the nearest integer required for the world population to triple in size.

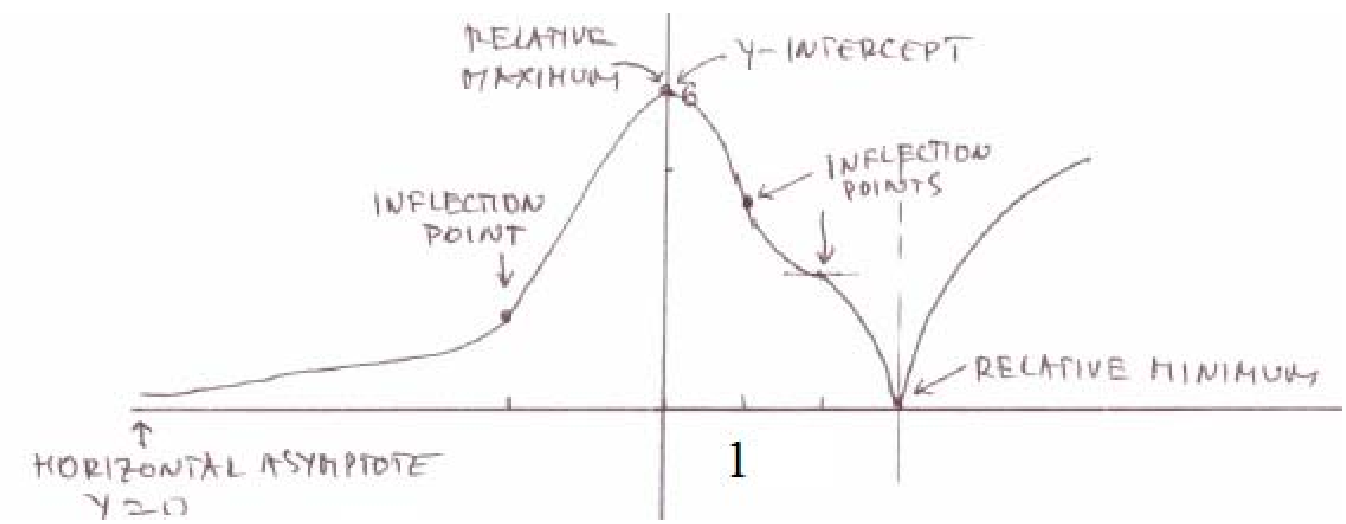
   We need to solve $3\times 5.3=5.3(1.02)^t$ for *t*.

$$3\times 5.3=5.3(1.02)^t$$
$$3=(1.02)^t$$
$$\ln(3)=\ln(1.02)^t=t\ln(1.02)$$
$$t=\frac{\ln(3)}{\ln(1.02)}\approx 55.478$$

   Therefore, the population will triple in size in approximately 55 years.

6. Sketch the graph of a function *f* such that a) to k) below hold. For full marks clearly and carefully label all intercepts, relative extrema, inflection points, and asymptotes. (Instructor Questions #9 Q1) **[7 marks]**

   a) Domain: $(-\infty,\infty)$.
   b) Continuous for all real numbers.
   c) Differentiable everywhere except at $x=3$.
   d) $f(0)=6$
   e) $\lim_{x\to-\infty} f(x)=0$
   f) $f'(0)=f'(2)=0$
   g) $f'(x)>0$ on $(-\infty,0)$ and $(3,\infty)$
   h) $f'(x)<0$ on (0, 2) and (2, 3)
   i) $\lim_{x\to3^+} f'(x)=\infty$ and $\lim_{x\to3^-} f'(x)=-\infty$
   j) $f''(x)>0$ on $(-\infty,-2)$ and (1, 2)
   k) $f''(x)<0$ on (−2, 1), (2, 3), and $(3,\infty)$.



7. You are given the function *f*, and its first and second derivatives: $f(x)=\dfrac{2x}{x^2+1}$,

$f'(x)=\dfrac{2\left(1-x^2\right)}{\left(x^2+1\right)^2}$, $f''(x)=\dfrac{4x\left(x^2-3\right)}{\left(x^2+1\right)^3}$. (textbook exercise 7.2 #72) **[9 marks]**

a) Determine the intervals of increase and decrease.

$f'(x)=0 \Rightarrow\ 2\left(1-x^2\right)=0 \Rightarrow\ x=\pm1$. So, the critical numbers are $x=\pm1$.

		-1		1	
$f'(x)$	-		+		-
$f(x)$	decr		incr		decr

Therefore, *f* is decreasing on $(-\infty,-1)\cup(1,\infty)$, and increasing on $(-1,1)$.

b) Determine the intervals of concave up and concave down.

$f''(x)=0 \Rightarrow\ 4x\left(x^2-3\right)=0 \Rightarrow\ x=0,\pm\sqrt{3}$.

		-sqrt(3)		0		sqrt(3)	
$f''(x)$	-		+		-		+
$f(x)$	c.d.		c.u.		c.d.		c.u.

Therefore, *f* is concave up on $\left(-\sqrt{3},0\right)\cup\left(\sqrt{3},\infty\right)$, and concave down on $\left(-\infty,-\sqrt{3}\right)\cup\left(0,\sqrt{3}\right)$.

c) Answer **T** (true) or **F** (false) in the boxes provided about the function *f*.

T f is an odd function.

T $\lim\limits_{x\to\pm\infty} f(x)=0$.

F $\lim\limits_{x\to1^-} f(x)=-\infty$ and $\lim\limits_{x\to1^+} f(x)=\infty$.

F f has a relative maximum at $x=0$.

T f has an absolute minimum at $x=-1$.

T f has an inflection point at $x=0$.

8. The demand equation for a product is $x+0.03p=12$ where p is the price in dollars per unit with $0 \le p \le 300$ and x is the quantity in thousands of units demanded. **[8 marks]**

a) Determine the elasticity of demand function $E(p)$ at price p.

$$\frac{d}{dp}(x+0.03p)=\frac{d}{dp}12$$

$$\frac{dx}{dp}+0.03=0 \qquad \text{and} \qquad \begin{aligned} x+0.03p&=12 \\ x&=12-0.03p \end{aligned}$$

$$\frac{dx}{dp}=-0.03$$

Then, $E(p)=-\frac{p}{x}\frac{dx}{dp}=-\frac{p}{12-0.03p}(-0.03)=\frac{0.03p}{12-0.03p}$.

b) Solve $E(p)=1$ for p.

$$\begin{aligned} E(p)&=1 \\ \frac{0.03p}{12-0.03p}&=1 \\ 0.03p&=12-0.03p \\ 0.06p&=12 \\ p&=200 \end{aligned}$$

c) Answer **T** (true) or **F** (false) in the boxes provided about the demand.

T	The demand is inelastic if $0 \le p < 200$.
F	The demand is inelastic if $200 < p \le 300$.
T	For $p=50$ an increase in the unit price will cause the revenue to increase.
T	For $p=50$ a decrease in the unit price will cause the revenue to decrease.

d) Continues from Question 8. If a price of \$40 is increased by ½ %, what is the approximate change in demand?

$$\Delta x \approx dx = \frac{dx}{dp} \cdot dp$$

$$p_1 = 40, p_2 = 40 \cdot 1.005 = 40.2, dp = 40.2 - 40 = 0.2$$

$$\frac{dx}{dp} = -0.03$$

So, $\Delta x \approx -0.03 \cdot 0.2 = -0.006$.

9. Suppose the quantity *x* of Super Grip radial tires made available each week in the marketplace is related to the unit-selling price by the equation $p - \frac{1}{2}x^2 = 48$, where *x* is measured in units of a thousand and *p* is in dollars. How fast is the weekly supply of Supper Grip radial tires being introduced into the marketplace when *x=6, p=66,* and the price per tire is decreasing at the rate of $3/week? (textbook exercise 4.4 #54) **[8 marks]**

Given $\frac{dp}{dt} = -3$ dollars/week.

Differentiating the given equation $p - \frac{1}{2}x^2 = 48$ implicitly w.r.t. *t* we get

$$\frac{d}{dt}\left(p - \frac{1}{2}x^2\right) = \frac{d}{dt}(48)$$

$$\frac{dp}{dt} - x\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{dp}{dt} \cdot \frac{1}{x}$$

When $x = 6, p = 66$, and $\frac{dp}{dt} = -3$ we have

$$\left.\frac{dx}{dt}\right|_{x=6,\, p=66} = (-3) \cdot \frac{1}{6} = -\frac{1}{2} = -0.5\,.$$

Therefore, the supply is decreasing at the rate of 0.5 thousand tires/week.

(or the supply is decreasing at the rate of 500 tires/week)

10. The Smith family amortizes a loan of \$500,000 for a new house by obtaining a 25 year mortgage with monthly payments at the nominal rate of 5.5 % compounded semiannually. **[10 marks]**

a) Calculate the nominal rate compounded monthly.

$$r_{monthly} = 12\left[\left(1+\frac{0.055}{2}\right)^{2/12} - 1\right] \approx 0.05438$$

b) Find the monthly payment.

$$P = 500{,}000 \qquad m = 12 \qquad i = \frac{r_{monthly}}{12} \approx 0.004532$$

$$t = 25 \qquad n = 25 \cdot 12 = 300$$

$$R = 500{,}000\left[\frac{1-(1.004532...)^{-300}}{0.004532...}\right]^{-1} \approx \$3051.96$$

c) Find the total interest charges.

$$\text{interest} = \$3051.96 \cdot 300 - \$500{,}000 = \$415{,}588.00$$

d) After 18 years the Smith family decides to pay off the loan. How much money is needed to pay off the balance of the loan?

After 18 years, there are 7 years remaining, i.e. $n = 7 \cdot 12 = 84$.

$$P = 3051.96\left[\frac{1-(1.004532...)^{-84}}{0.004532...}\right] \approx \$212{,}818.93$$