Simon Fraser University, Department of Mathematics, Burnaby Campus

Math 157, Summer 2010

Midterm 2

July 14, 2010, 11:30 a.m.–12:20 p.m., AQ 3182

Last Name (please print): ______________________

First Name (please print): ______________________

Student Number: ______________________

SFU Email (please print): ______________________@sfu.ca

Instructor: Roland Wittler

Instructions

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.
2. Fill in the above box.
3. This exam contains 7 pages with a total of 5 questions. Once the exam begins please check to make sure your exam is complete.
4. SHOW ALL YOUR WORK!
5. If you run out of space in a problem, use the space on the back of the previous page and clearly indicate where the solution continues.
6. Only scientific, non-programmable calculators with no differentiation and integration capabilities are allowed.
7. No book, paper, or device, other than the usual writing instruments, this booklet and an acceptable calculator, shall be within reach of a student during the examination.
8. During the examination, speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.
9. Try your Best!

Do not write in this table.	
Question	**Marks**
1	/6
2	/6
3	/4
4	/6
5	/8
Total	**/30**

Question 1: Fill the gaps in the following statements. (6 marks)

(a) The increment Δy measures the ________________________ change in y, whereas the differential dy measures the ________________________ change in y.

[actual, approximate]

(b) The graph of the function $f(x) = b^x$ has the _____-axis as a ________________________ asymptote.

[x, horizontal]

(c) The graph of the function $f(x) = \log_b x$ is decreasing on its domain if _______________.

[$0 < b < 1$]

(d) If $f(x) = e^{g(x)}$, than $f'(x) =$ _______________.

[$g'(x)e^{g(x)}$]

(e) Pythagorean Identity: $\sin^2\theta + \cos^2\theta =$ __________ .

[1]

(f) Converting Degrees to Radians: $360°=$ __________ radians.

[2π]

Question 2: Linear Approximation. (6 marks)

Consider the following function:

$$f(x) = 2 + \frac{1}{x}\,, x > 0\,.$$

(a) Find the linear approximation $L(4)$ of $f(4)$ at $a = 2$.

$$\begin{aligned}
f(a) &= f(2) = 2 + \frac{1}{2} = 2.5 \\
f'(x) &= -x^{-2} = -\frac{1}{x^2} \\
f'(a) &= f'(2) = -\frac{1}{2^2} = -0.25 \\[1em]
L(x) &= f(a) + f'(a)\,(x - a) \\
&= 2.5 - 0.25(x - 2) \\
&= 3 - 0.25x \\[1em]
L(4) &= 3 - 0.25 \cdot 4 = 3 - 1 = 2
\end{aligned}$$

(b) Would you expect the linear approximation of $f(4)$ at $a = 3$ to be more accurate? Explain your answer without computing $L(4)$ for $a = 3$.

Yes, I expect it to be more accurate. Since $x = 4$ is much closer to $a = 3$ than $a = 2$ (and $f(x)$ is decreasing for all $x > 0$), the actual value of $f(4)$ is closer to the linear approximation at $a = 3$ than the linear approximation at $a = 2$.

Question 3: The Newton-Raphson Method. (4 marks)

Estimate the value of $\sqrt[3]{40}$ using the Newton-Raphson method for the function $f(x) = x^3 - 40$.

(a) Give the formula for x_{n+1} with respect to x_n in its explicit form (without f or f').

$$\begin{aligned} f'(x) &= 3x^2 \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{{x_n}^3 - 40}{3{x_n}^2} \end{aligned}$$

(b) For the initial guess $x_1 = 3$, compute x_2, x_3 and x_4.
Give the values rounded to 5 decimals, but calculate with highest possible accuracy.

$$\begin{aligned} x_1 &= 3 \\ x_2 \Bigg[&= x_1 - \frac{{x_1}^3 - 40}{3{x_1}^2} \\ &= 3 - \frac{3^3 - 40}{3 \cdot 3^2} \Bigg] \\ &\approx 3.48148 \\ x_3 &\approx 3.42103 \\ x_4 &\approx 3.41995 \end{aligned}$$

Question 4: Exponential Functions as Mathematical Models. (6 marks)

To brew a pot of tea, boiling water is filled into a pot containing a few tea bags. The temperature of the cooling water can be described by the following exponential decay function:

$$T(t) = A\,e^{-kt} + 20\,,$$

where A and k are constants, the temperature T is measured in °C, and the elapsed time t is measured in minutes.

(a) Let $t = 0$ correspond to the time point where the boiling water (100°C) is filled into the pot. Compute the constant A.

$$\begin{aligned} T(0) &= a\,e^0 + 20 = a + 20 = 100 \\ \Leftrightarrow a &= 80 \end{aligned}$$

(b) When the tea bags are removed after 3 minutes, the temperature is 75°C. Compute the decay constant k.

$$\begin{aligned} T(3) &= 80\,e^{-3k} + 20 = 75 \\ \Leftrightarrow e^{-3k} &= 55/80 \\ \Leftrightarrow -3\,k &= \ln(55/80) \\ \Leftrightarrow k &= -\frac{\ln(55/80)}{3} \approx 0.124898 \end{aligned}$$

(c) Compute the time t at which the tea is cooled down to a comfortable temperature of 40°C.

$$\begin{aligned} T(t) &= 80\,e^{\frac{\ln(55/80)}{3}t} + 20 = 40 \\ \Leftrightarrow e^{\frac{\ln(55/80)}{3}t} &= 20/80 = 1/4 \\ \Leftrightarrow \frac{\ln(55/80)}{3}t &= \ln(1/4)\ [= \ln 1 - \ln 4 = -\ln 4] \\ \Leftrightarrow t &= 3\,\frac{\ln(1/4)}{\ln(55/80)} \approx 11.09943 \end{aligned}$$

[Or using the approximate value for k from the beginning.]

Question 5: Find the derivatives as indicated below. Do not simplify unless otherwise noted. (8 marks)

(a) $f(x) = \sin(\ln(x))$, $f'(x)$

$$f'(x) = \frac{\cos(ln(x))}{x}$$

(b) $y = x^{(2x^3)}$, give $\dfrac{dy}{dx}$ in terms of x.

$$\begin{aligned} y &= x^{(2x^3)} \\ \Leftrightarrow \ln(y) &= \ln\left(x^{(2x^3)}\right) = 2x^3 \ln(x) \\ \Rightarrow \frac{1}{y}\frac{dy}{dx} &= 6x^2 \ln(x) + 2x^3 \frac{1}{x} \\ \Leftrightarrow \frac{dy}{dx} &= y \left(6x^2 \ln(x) + 2x^2\right) \\ &= x^{(2x^3)} \left(6x^2 \ln(x) + 2x^2\right) \end{aligned}$$

(c) $g(\theta) = \cos(\theta)\, e^{\theta}\,,\ \dfrac{dg}{d\theta}$

$$\frac{dg}{d\theta} = -\sin(\theta)e^{\theta} + \cos(\theta)e^{\theta}\ \left[= e^{\theta}(\cos(\theta) - \sin(\theta))\right]$$

(d) $2x^3 + 4y^2 = 5\,,\ \dfrac{dy}{dx}$

$$\begin{aligned} \Rightarrow 2\frac{d}{dx}(x^3) + 4\frac{d}{dx}(y^2) &= \frac{d}{dx}(5) \\ \Leftrightarrow 6x^2 + 8y\frac{dy}{dx} &= 0 \\ \Leftrightarrow \frac{dy}{dx} &= -\frac{6x^2}{8y} \end{aligned}$$