Simon Fraser University, Department of Mathematics, Burnaby Campus

Math 157, Summer 2010

Final Exam

August 14, 2010, 12:00–15:00 p.m.

Last Name (please print): _____

First Name (please print): _____

Student Number: _____

SFU Email (please print): _____ @sfu.ca

Instructor: Roland Wittler

Instructions

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.
2. Fill in the above box.
3. This exam is printed DOUBLE-SIDED and contains 22 pages with a total of 12 questions. Once the exam begins please check to make sure your exam is complete.
4. SHOW ALL YOUR WORK!
5. If you run out of space in a problem, continue on page 21.
6. Only scientific, non-programmable, non-graphical calculators with no differentiation and integration capabilities are allowed.
7. No book, paper, or device, other than the usual writing instruments, this booklet and an acceptable calculator, shall be within reach of a student during the examination.
8. During the examination, speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.
9. Try your Best!

GOOD LUCK!

Do not write in this table.			
Qtn.	**Marks**	**Qtn.**	**Marks**
1	/10	7	/6
2	/9	8	/12
3	/12	9	/8
4	/7	10	/6
5	/7	11	/7
6	/7	12	/9
Total			**/100**

Question 1: Fill the gaps in the following statements. (10 marks)

(a) The graph of the function $f(x+2)+3$ can be found by shifting the graph of $f(x)$ 2 units _____ and 3 units _____ .

(b) The graph of the inverse function $f^{-1}(x)$ corresponds to the graph of the function $f(x)$ reflected in the line with the equation _____ .

(c) If the graph of the function $f(x)$ intercepts the y-axis at $y = 2$ and $f'(0) = 3$, then the equation of the tangent line to the graph of f at $x = 0$ is _____ .

(d) The linear approximation of $f(x)$ near a is given by

$$f(x) \approx L(x) = \_\_\_\_\_ .$$

(e) For a given function $f(x)$, the Newton-Raphson method can be used to approximate a solution for the equation _____ .

(f) If $y = \log_a(x)$, then ___ is the exponent to which the base ___ has to be raised to equal the value of ___ .

(g) Laws of exponents: $b^{x+y} =$_____, and $(ab)^x =$_____ .

(h) A quantity is described by the function $Q(t) = 20 \cdot 2^{3t}$, where t is measured in hours. Then the doubling time is _____ minutes.

(i) Among the trigonometric functions $\sin\theta$, $\cos\theta$ and $\tan\theta$, only _____ is positive for $\theta = \frac{2\pi}{3}$, and only _____ is positive for $\theta = \frac{8\pi}{9}$.

(j) If the function $f(x)$ is continuous on the interval $[a, b]$, c and d are the only critical numbers of f in $[a, b]$, and $f(d) < f(a) < f(c) < f(b)$, then f has an absolute minimum at _____, and an absolute maximum at _____ .

Question 2: Limits and Continuity. (9 marks)

Consider the following function $f(x)$ with parameter a:

$$f(x) = \begin{cases} e^{-x} + 2a & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ \dfrac{x^2 - 5}{x^2 + 5} & \text{for } x > 0 \end{cases}$$

In (a)–(d), evaluate the given limit, if it exists. If a limit does not exist, then write $+\infty$, $-\infty$, or DNE, whichever is most informative.

(a) Limit of f as x approaches negative infinity.

(b) Limit of f as x approaches positive infinity.

(c) Left-hand limit of f as x approaches 0 from the left.

(d) Right-hand limit of f as x approaches 0 from the right.

(e) Find a value of a such that the limit of f as x approaches 0 exists.

(f) Verify whether f is continuous at $x = 0$ for the value of a you found in (e). Check all necessary conditions for continuity! (If you did not find any value for a in (e), use $a = 0$.)

Question 3: Derivatives. (12 marks)

Find the derivatives as indicated below. You do not have to simplify.

(a) $g(\theta) = \dfrac{2\theta^3 + \theta}{\cos\theta}$, $\dfrac{dg}{d\theta}$

(b) $f(x) = 2^x \sqrt{x}$, $f'(x)$

(c) $P(x) = (\ln x)^2$, find the second derivative of $P(x)$.

(d) $x^2y = y^2x + 1$, use *implicit differentiation* to find $\dfrac{dy}{dx}$.

(e) $y = x^x$ $(x \neq 0)$, use *logarithmic differentiation* to find $\left.\dfrac{dy}{dx}\right|_{x=1}$.

Question 4: Implicit Differentiation and Related Rates. (7 marks)

The company *Pisa Construction Ltd.* builds a tower of height $h = 50$ metres, but then the tower inclines: The angle θ between the tower and the ground decreases from originally $90°$ by $0.1°$ per year. Let a denote the altitude of the top of the tower measured in metres.

(a) Sketch a diagram of the leaning tower and label pertinent parts with the given variables.

(b) Give an equation that relates θ, a and h.

(c) How fast is a decreasing when $\theta = 89°$?
(Use a full sentence to answer the question and do not forget to include units.)

Question 5: Curve Sketching. (7 marks)

Consider the function $f(x) = \dfrac{e^x}{x}$ with $f'(x) = \dfrac{e^x(x-1)}{x^2}$ and $f''(x) = \dfrac{e^x(x^2-x+2)}{x^3}$.

(a) Find the domain.

(b) Find all horizontal asymptotes.

(c) Find all vertical asymptotes.

(d) Find all intervals of increase and decrease.

(e) Give all points where f has a relative extremum and specify whether it is a relative maximum or relative minimum.

(f) Given that $f'(x)$ has no critical numbers on its domain $\mathbb{R}\backslash\{0\}$, give all intervals of concave up and concave down.

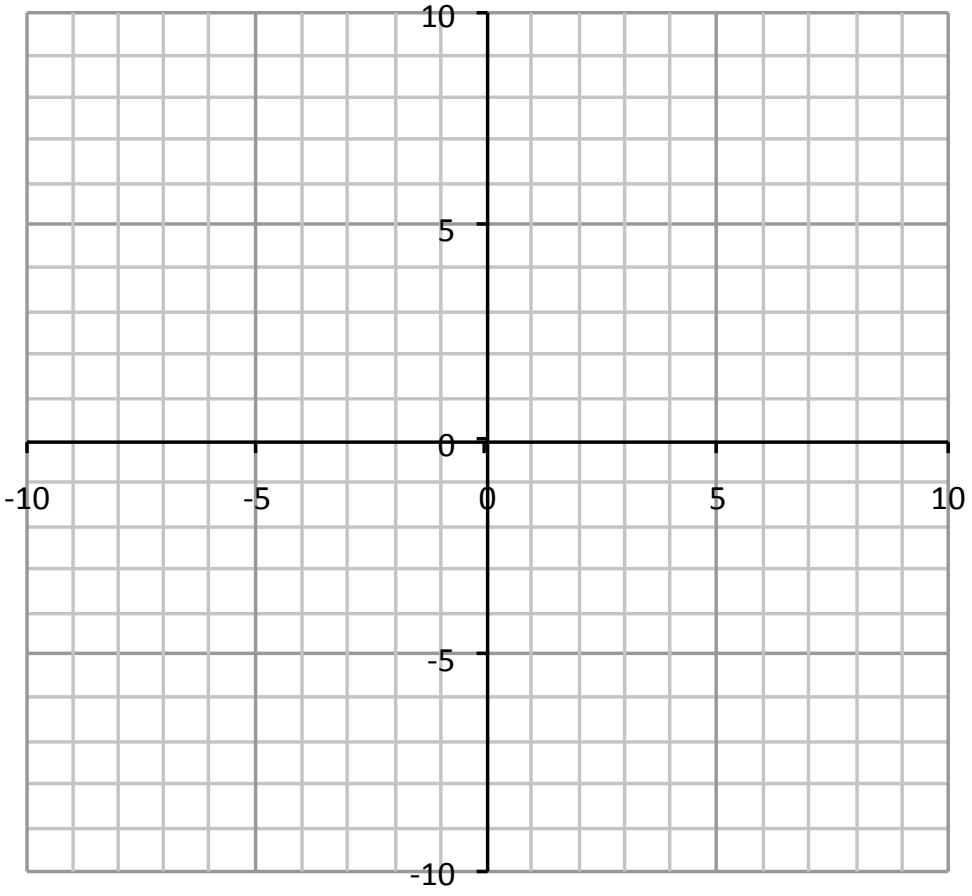
Question 6: Graphing. (7 marks)

(a) Sketch the graph of a function f with the following properties in the coordinate system given below.

- Intercepts $f(0) = -7$, $f(-3) = 0$ and $f(5) = 0$.
- Horizontal asymptote $y = 3$ for $x \to -\infty$.
- Horizontal tangent line at $x = 5$.
- $f'(x) < 0$ for $x < 0$, and $f'(x) \geq 0$ for $x > 0$.
- $f''(x) < 0$ for $x < 0, 0 < x < 5$, and $f''(x) \geq 0$ for $x > 5$.

(b) Sketch the derivative of f in the same coordinate system.

Indicate pertinent points and lines, and label the graphs!



Question 7: Elasticity of Demand. (6 marks)

The management of *Cookagoose* determines the relation of the demanded quantity of their product *Cook-O-Matics* and its unit price. They find that the Elasticity of demand is

$$E(p) = \frac{p}{2(1001 - p)}\,.$$

(a) Is the demand elastic or inelastic when the price is \$858? (Show your work.)

(b) Mark the correct statement:

If the price is increased from \$858 by 1%, the revenue will

- ☐ decrease by approximately 0.03%.
- ☐ increase by approximately 0.03%.
- ☐ decrease by approximately 1/3%.
- ☐ increase by approximately 1/3%.
- ☐ decrease by approximately 3%.
- ☐ increase by approximately 3%.
- ☐ decrease by approximately 300%.
- ☐ increase by approximately 300%.

(c) For which price do we have unitary demand?

Question 8: Optimization. (12 marks)

Little Susan wants to upgrade her pocket money by selling self-made, freshly squeezed lemonade on a sunny summer day. She estimates the cost in dollars for producing x lemonades to be: $C(x) = 0.0625x^2 + 0.2x + 4$.

(a) Find the average cost function $\overline{C}(x)$ and compute the number of lemonades x in $(0, \infty)$ that results in an absolute minimum of the average production cost.

(b) Susan further estimates that the number of lemonades demanded is related to the price by the demand equation: $p = 4 - 0.04x$ for $0 \leq x \leq 100$. Find the revenue function $R(x)$.

(c) Compute the number of lemonades x in $[0, 100]$ where the revenue has its absolute maximum.

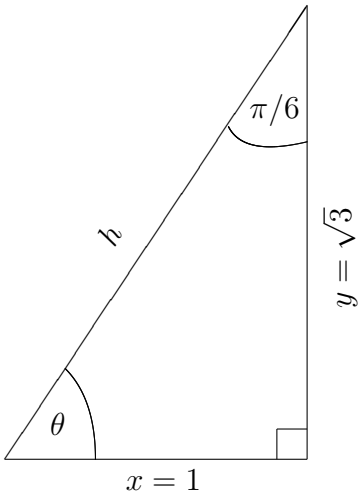
(d) Find the profit function $P(x)$ and determine whether Susan makes a profit for the values found in (a) and (c).

Question 9: Trigonometric Functions. (8 marks)

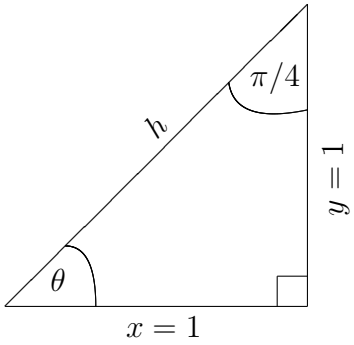
Each of the following eight statements holds for one of the four triangles shown below. Assign each statement to its corresponding triangle by placing the **label** on the given line. Note that some triangles go with more than one statement, but each statement only goes with one triangle.

(A) $h = 6$ (B) $\sin\theta = 2/h$ (C) $\theta = \pi/3$ (D) $\tan\theta = \dfrac{2}{x}$

(E) $h = \sqrt{2}$ (F) $\theta = \cos^{-1}(2/h)$ (G) $\operatorname{cotan}\theta = \tan\theta$ (H) $\theta = 45°$



statement(s):_____



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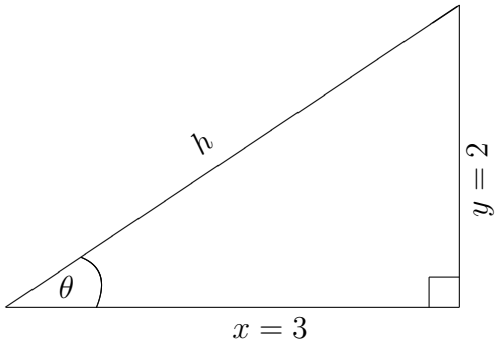

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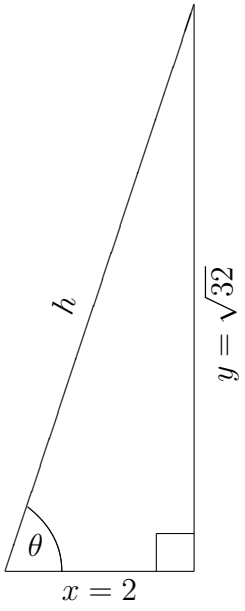

statement(s):_____

Question 10: Simple Interrest. (6 marks)

(a) Find the future value after two years if \$3,000 are invested at a simple interest rate of 3% per year.

(b) Find the simple interest rate needed for a loan of \$20,000 to grow to an amount of \$30,000 in 15 years. Express the rate as a percentage accurate to two decimal places.

(c) How many years did it take for a loan of \$10,000 to accumulate a simple interest of \$1,000 at an annual interest rate of 4.5%? Round your result to two decimal places.

Question 11: Compound Interrest. (7 marks)

Ben wants to save money to buy a small sailboat when he retires. He plans to deposit a certain amount of money for 40 years at 2% interest rate per year compounded quarterly.

(a) How much will he have to invest now to be able to buy a boat for $30,000? Round your result to the nearest cent.

(b) How much interest will Ben earn? Round your result to the nearest cent.

(c) Ben's current finances, together with a little help of Grandma, allow him to invest \$15,000. After how many years will the future value be sufficient to buy a boat for \$30,000? Round your result to two decimal places.

Question 12: Decreasing Annuity. (9 marks)

The Pearsons want to borrow money to buy a house. Their bank offers a loan at 6% interest compounded semiannually. Checking their financial situation, they estimate that they are able to pay $10,000 every six months. They want to pay off their balance within 30 years.

Recall that: $P = R\left[\dfrac{1-(1+i)^{-n}}{i}\right]$. Round all your final results to the nearest cent.

(a) What is the principle the Pearsons can afford under the given conditions?

(b) What will be the total amount they pay for the house, and what is the total amount of interest paid?

(c) How much do the Pearsons owe the bank after 28 years?

(d) After 28 years, the Pearsons decide to pay off the mortgage with a lump sum payment. What is the total amount of interest paid in this case?

You can use this page and its back, if you need more room for any of your answers.

Important — otherwise this page will not be marked:

- On the previous pages: Clearly label an incomplete answer with a $\divideontimes$ to indicate that your answer continues on this page!
- On this page: Clearly label your work with the number and letter of the corresponding problem!

Important — otherwise this page will not be marked:

- On the previous pages: Clearly label an incomplete answer with a ✳ to indicate that your answer continues on this page!
- On this page: Clearly label your work with the number and letter of the corresponding problem!