Simon Fraser University
Department of Mathematics
Burnaby Campus

MATH 157 - D100 Spring09 Calculus I for the Social Sciences

Midterm 2 – Version 1

March 11th 2009, 11:30–12:20

Last Name (please print):	**Solutions**
First Name (please print):	____________________
SFU Email ID:	____________@sfu.ca
Student number:	____________________
Signature: **(do not sign before your ID is checked)**	____________________
Instructor:	Y. van Gennip

Instructions:

1. **Do not open this booklet until told to do so.**
2. Fill in the above box. Please use the name under which you are registered.
3. This exam contains 7 pages with a total of 5 questions. Once the exam begins please check to make sure your exam is complete.
4. **Show all your work! Justify your answer unless it is specifically stated that you do not need to.**
5. If you run out of space in a problem, use the space on the back of the previous page and clearly indicate where the solution continues.
6. **Only** scientific, non-programmable calculators with no graphing, differentiation, and integration capabilities are allowed.
7. No book, paper, or device, other than the usual writing instruments, this booklet, and an acceptable calculator shall be within reach of a student during the examination.
8. During the examination speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.
9. Try your best!

Do not write in this table!	
Question	**Marks**
1	**/4**
2	**/3**
3	**/6**
4	**/8**
5	**/11**
Total	**/32**

1. Answer the following questions with "true" or "false". No explanation is necessary. **[1/2 mark each = 4 marks]**

   (a) If the cost for producing x units is given by $C(x)$, then $\frac{C(x)}{x}$ is the marginal cost.

   False, $C'(x)$ is the marginal cost.

   (b) If there is a value $x = c$ such that $f'(c) = 0$, then the graph of the function f is a horizontal line.

   False, the tangent line to the graph in $(c, f(c))$ is horizontal, not necessarily the whole graph.

   (c) $\sin(x - y) = \sin x \cos y - \cos x \sin y$, for any real numbers x and y.

   True

   (d) The function $f(x) = e^{-(x-1)^2}$ is increasing on the interval $(-\infty, 1)$.

   True, since $f'(x) = 2(1 - x)e^{-(x-1)^2}$.

   (e) If the function f is continuous on the interval $[a, b]$ and $f(c)$ is the absolute maximum of f on this interval, then either $c = a$, $c = b$, or c is a critical number.

   True

   (f) If $f(x) = g(h(x))$ for some functions g and h, then $f'(x) = g'(x)h'(x)$.

   False, by the chain rule $f'(x) = g'(h(x))h'(x)$.

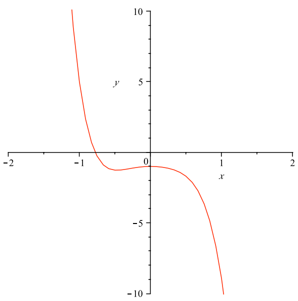
   (g) If $(c, f(c))$ is an inflection point for the function f, then $f''(c) = 0$ or $f''(c)$ does not exist.

   True

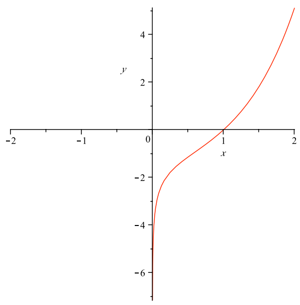
   (h) If f is a polynomial of degree n, where $n \geq 4$, then $f^{(4)}$ is a polynomial of degree at most $n - 4$.

   True, because by the power rule $(x^n)' = nx^{n-1}$.

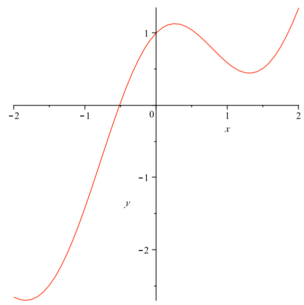
2. Under (a), (b), and (c) graphs of functions are given. For each of these find the graph of the corresponding derivative function under (1)-(6). Write the number next to the graph of the function. There are three extra graphs given that do not correspond to any of the graphs under (a)-(c). An explanation is not necessary. **[1 mark each = 3 marks]**



(a)

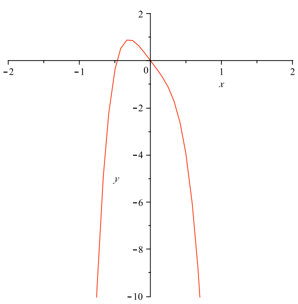


(b)

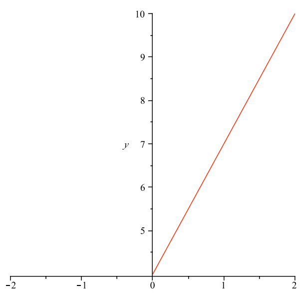


(c)

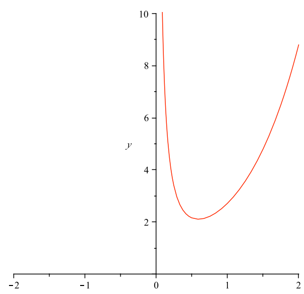
a1, b3, c4



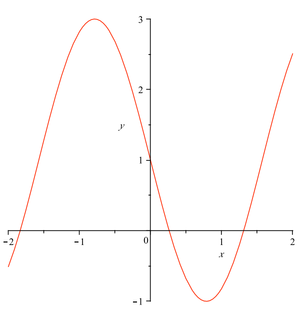
(1)



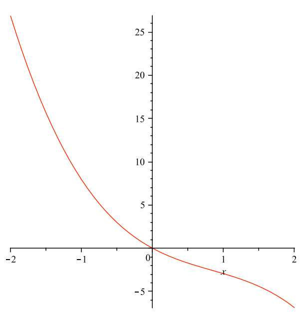
(2)



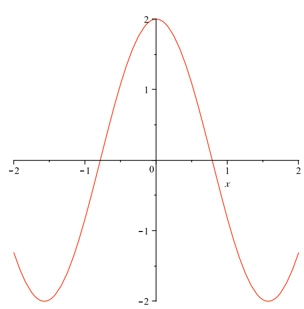
(3)



(4)



(5)



(6)

3. Compute the derivatives of the given functions. You don't need to simplify the answer. [**2 marks each = 6 marks**]

(a) $f(x) = \tan\left(\frac{x^2-3}{x+1}\right)$.

$f'(x) = \sec^2\left(\frac{x^2-3}{x+1}\right)\frac{2x(x+1)-(x^2-3)}{(x+1)^2} = \sec^2\left(\frac{x^2-3}{x+1}\right)\frac{x^2+2x+3}{(x+1)^2}$

(b) $g(x) = \frac{1}{\sqrt[3]{x^2}} - \left(\ln(1-x^2)\right)^2$

$g'(x) = -\frac{2}{3}x^{-\frac{5}{3}} - 2\ln(1-x^2)\frac{1}{1-x^2}(-2x)$
$= -\frac{2}{3x\sqrt[3]{x^2}} + \frac{4x}{1-x^2}\ln(1-x^2)$

(c) $h(x) = 71xe^x + \frac{98}{99}$

$h'(x) = 71e^x + 71xe^x = 71(1+x)e^x$.

4. The owner of a coffee place has computed that the average cost in dollars of making x cups of coffee during a day is given by the function $\overline{C}(x) = \frac{\sqrt{540x+6300}}{x}$. He sells the coffee for \$3 per cup. [**8 marks**]

   (a) Compute the marginal cost function.

   The cost function is given by

   $$C(x) = x\overline{C}(x) = \sqrt{540x + 6300} = 3\sqrt{60x + 700} = 6\sqrt{15x + 175}$$

   and thus the marginal cost function is

   $$C'(x) = \frac{540}{2\sqrt{540x + 6300}} = \frac{270}{\sqrt{540x + 6300}} = \frac{90}{\sqrt{60x + 700}} = \frac{45}{\sqrt{15x + 175}}.$$

   (b) Give a function P which expresses the profit if x cups of coffee are made and sold.

   Revenue: $R(x) = 3x$.
   Profit: $P(x) = R(x) - C(x) = 3x - \sqrt{540x + 6300}$.

   (c) Find the break-even quantity.

   The break-even quantity is x such that $P(x) = 0$, or equivalently $R(x) = C(x)$.

   $$\begin{aligned} 3x = \sqrt{540x + 6300} &\Rightarrow 9x^2 = 540x + 6300 \\ &\Leftrightarrow x^2 - 60x - 700 = 0 \\ &\Leftrightarrow x = \frac{1}{2}\left(60 \pm \sqrt{3600 + 2800}\right) = 30 \pm 40. \end{aligned}$$

   Therefore the break-even quantity is 70 cups of coffee.

   (d) For which number of cups of coffee sold is the profit minimal? Remember that the coffee place can only sell an integer number of cups (so no fractions or decimals as answer). What is the profit or loss in this case? Give your answer accurate up to dollar cents.

   First we need to find the critical numbers of P.

   $$P'(x) = 3 - \frac{45}{\sqrt{15x + 175}} = \frac{3\sqrt{15x + 175} - 45}{\sqrt{15x + 175}}.$$

   So there are no numbers x in the domain of P such that $P'(x)$ does not exist. Furthermore

   $$\begin{aligned} P'(x) = 0 &\Leftrightarrow \sqrt{15x + 175} = 15 \\ &\Rightarrow 15x + 175 - (15)^2 = 15x - 50 = 0 \\ &\Rightarrow x = \frac{50}{15} = \frac{10}{3}. \end{aligned}$$

The first derivative test, i.e. computing $P'(x)$ for an $x < \frac{10}{3}$ and for an $x > \frac{10}{3}$ shows that $\left(\frac{10}{3}, P(\frac{10}{3})\right)$ is a local or relative minimum.

Since coffee can only be sold in an integer number of cups, the minimum profit is reached for either $x = 3$ or $x = 4$. $P(3) = 9 - 12\sqrt{55} \approx -79.99$ and $P(4) = 12 - 6\sqrt{235} \approx -79.98$. The minimum profit, a loss of \$79.99, is attained at 3 cups of coffee.

5. Given is the function

$$f(x) = \frac{2x}{x^2 - 1}.$$

The first and second derivative are given by

$$f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2}, \qquad f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}.$$

[11 marks]

(a) What is the domain of f?

Dom $f = \mathbb{R} \setminus \{\pm 1\}$,

(b) Compute any x- and y-intercepts of f.

$$f(x) = 0 \Leftrightarrow x = 0, \qquad f(0) = 0.$$

So the only x- and y-intercept is the point $(0, 0)$.

(c) Find all horizontal and vertical asymptotes of f.

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x}{x^2} \frac{2}{1 - \frac{1}{x^2}} = 0,$$

so $y = 0$ is a horizontal asymptote.

$$\begin{aligned} \lim_{x \to -1^-} f(x) &= -\infty, \\ \lim_{x \to -1^+} f(x) &= \infty, \\ \lim_{x \to 1^-} f(x) &= -\infty, \\ \lim_{x \to 1^+} f(x) &= \infty, \end{aligned}$$

so there are vertical asymptotes at $x = \pm 1$.

(d) Find the intervals on which f is increasing and decreasing. Find any relative extrema of f and identify them as maxima or minima.

First find the critical numbers of f and the numbers x for which $f(x)$ does not exist. $f(x)$ does not exist for $x = \pm 1$. There are no numbers x in the domain of f such that $f'(x)$ does not exist. There are also no x such that $f'(x) = 0$.
The first derivative test shows then that $f'(x) < 0$ for all x in the domain of f and thus f is decreasing everywhere on its domain. There are no relative extrema.

Question 5 continues on the following page.

(e) Find all intervals where f is concave up or concave down and all inflection points.

First find all critical numbers of f' and numbers x where f does not exist. $f(x)$ does not exist for $x = \pm 1$. There are no numbers x in the domain of f such that $f''(x)$ does not exist. Furthermore $f''(x) = 0 \Leftrightarrow x = 0$.

The concavity test shows that $f''(x) < 0$ for $x \in (-\infty, -1)$ and for $x \in (0, 1)$, so f is concave down(wards) on these intervals. Furthermore $f''(x) > 0$ for $x \in (-1, 0)$ and $x \in (1, \infty)$ and therefore f is concave up(wards) on these intervals. $f(0) = 0$ and thus $(0, 0)$ is an inflection point. At $x = -1$ and $x = 1$ there are no inflection points, because these numbers are not in the domain of f.

(f) Use the information you have found to sketch the graph of f in the coordinate system below.