Simon Fraser University
Department of Mathematics
Burnaby Campus

MATH 157 - D100 Spring09 Calculus I for the Social Sciences

Final Exam

April 11th 2009, 8:30–11:30

Last Name (please print):	Solutions
First Name (please print):	______________________
SFU Email ID:	______________@sfu.ca
Student number:	______________________
Signature: **(do not sign before your ID is checked)**	______________________
Instructor:	Y. van Gennip

Instructions:

1. **Do not open this booklet until told to do so.**
2. Fill in the above box. Please use the name under which you are registered.
3. This exam contains 16 pages with a total of 12 questions. Once the exam begins please check to make sure your exam is complete.
4. **Show all your work! Justify your answer unless it is specifically stated that you do not need to.**
5. If you run out of space in a problem, use the space on the back of the previous page and clearly indicate where the solution continues.
6. **Only** scientific, non-programmable calculators with no graphing, differentiation, and integration capabilities are allowed.
7. No book, paper, or device, other than the usual writing instruments, this booklet, and an acceptable calculator shall be within reach of a student during the examination.
8. During the examination speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.
9. Try your best!

Do not write in this table!						
Question	1	2	3	4	5	6
Marks	**/16**	**/4**	**/8**	**/9**	**/6**	**/6**
Question	7	8	9	10	11	12
Marks	**/15**	**/8**	**/7**	**/9**	**/6**	**/6**
Total	**/100**					

1. Determine if the following statements are true or false. No explanation is necessary. **1 mark each = 16 marks**

   (a) Newton's method is a method to compute limits.

   F, it is a method to find the roots of a function.

   (b) If $f(c)$ exists, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$, then the function f is continuous at $x = c$.

   T

   (c) If demand for a product is inelastic, then an increase in the price of the product leads to an increase in the total revenue.

   T

   (d) If a function is concave up on an open interval, then the function is increasing on that interval.

   F, a function can both increase and decrease in a concave up way.

   (e) If $f(c)$ is a relative minimum of the function f, then $f(x) \geq f(c)$ for all x in the domain of f.

   F, for a relative minimum this inequality only needs to hold in an open interval which contains the number c.

   (f) A function that is continuous on a closed interval has an absolute maximum and an absolute minimum on that interval.

   T, this is the statement of the Extreme Value Theorem.

   (g) The derivative of a function f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x+h)}{h}.$$

   F, this limit gives $-f'(x)$.

   (h) The present value of an annuity of payments of R dollars each, made at the end of each period for n consecutive interest periods at a rate of interest i per period is $P = R\left(\frac{1-(1+i)^{-n}}{i}\right)$ dollars.

   T

Question 1 continues on the following page.

(i) The economic lot size is the is the number of products that should be produced in each batch to minimize the total cost of producing and storing the products.

T

(j) The graph of the function $f(x) = \sin x$ is concave up if and only if $f(x) < 0$.

T. The second derivative of f is $f''(x) = -\sin x = -f(x)$, so if $f(x) < 0$, then $f''(x) > 0$ and the graph of f is concave up.

(k) A function is a rule that assigns to each element from one set at least one element from another set.

F, a function assigns exactly one element from the range to each element of the domain.

(l) For $a > 0$, $a \neq 1$, and $x > 0$ the expression $y = \log_a x$ means $a^y = x$.

T

(m) Let f be a function with domain $D_f = \mathbb{R} \setminus \{5\}$ then f has a vertical asymptote at $x = 5$.

F, $x = 5$ is a candidate location for a vertical asymptote, but there does not need to be one. To check whether or not there is, you have to compute the limits $\lim_{x \to 5^-} f(x)$ and $\lim_{x \to 5^+} f(x)$.

(n) For a linear function the instantaneous rate of change is the same at each number in its domain.

T, the derivative function of a linear function is constant.

(o) Let f be a function with domain $D_f = \mathbb{R}$. If $f'(x) < 0$ for all $x < 4$, $f'(x) > 0$ for all $x > 4$, and $f(4) = 600,000$, then $600,000$ is the absolute minimum for f.

T, the function f is decreasing on $(-\infty, 4)$ and increasing on $(4, \infty)$ and $f(x)$ is defined at $x = 4$. Therefore the absolute minimum is attained at $x = 4$ with value $f(4) = 600,000$.

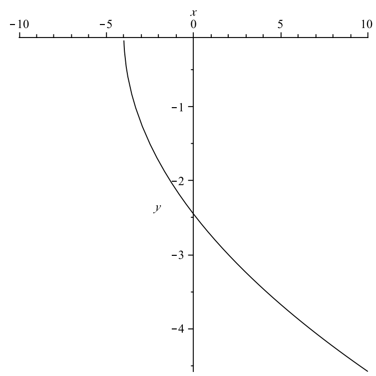
(p) If P dollars is invested at a yearly rate of interest r per year, compounded m times per year for t years, the compound amount is $A = P\left(1 + \frac{r}{m}\right)^{tm}$ dollars.

T

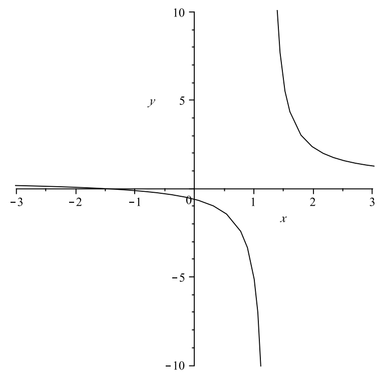
2. Match each of the graphs below with one of the following functions.

$$f(x) = x^9 - \frac{15}{8}x^7 + \frac{273}{256}x^5 - \frac{205}{1024}x^3 + \frac{9}{1024}x,$$
$$g(x) = -\sqrt{\frac{3}{2}x + 6},$$
$$h(x) = x^7 - \frac{67}{61}x^6 - 9x^5 + x^4 + x^3 + \frac{691}{1313}x,$$
$$k(x) = \frac{6}{5}\tan\left(\frac{x-3}{2}\right),$$
$$l(x) = \frac{2x+3}{4x-5}.$$

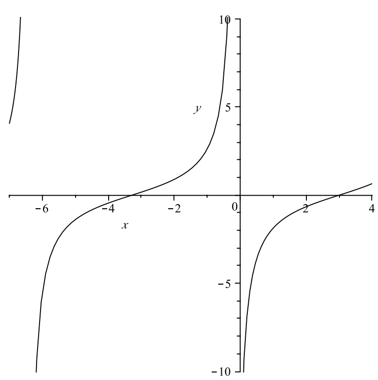
Note that there is one more function given than graphs. No explanation is necessary. [**1 mark each = 4 marks**]

(a)


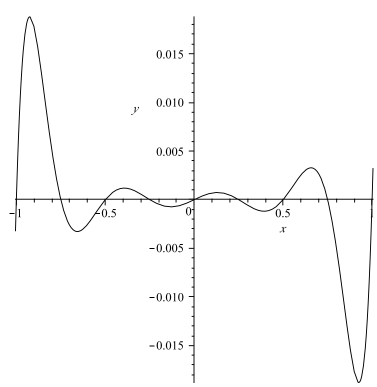
$y = g(x)$

(b)


$y = l(x)$

(c)


$y = k(x)$

(d)


$y = f(x)$

(a), (b), and (c) can be recognized by their general shape. To see that (d) is a graph of f and not h, we use that a polynomial of degree n can have at most $n-1$ critical points (or "turning points" as they were called in Chapter 2.3). Since the graph in (d) has 8 critical points, it cannot be the graph of h, since h is a polynomial of degree 7. Another way to see this, is via the number of x-intercepts. The graph has 9 x-intercepts and a polynomial of degree n has at most n roots, so once again we see that the graph cannot be of a polynomial of degree 7.

3. Compute the following limits. **2 marks each = 8 marks**

(a) $\lim\limits_{x\to 2} \frac{x^2+2x-8}{x-2}$.

$\ldots = \lim\limits_{x\to 2} \frac{(x-2)(x+4)}{x-2} = \lim\limits_{x\to 2} x+4 = 6.$

(b) $\lim\limits_{x\to \frac{3}{2}\pi} e^{\frac{\cos x}{x}}$.

$$\ldots = \exp\left(\frac{\lim\limits_{x\to\frac{3}{2}\pi} \cos x}{\lim\limits_{x\to\frac{3}{2}\pi} x}\right) = e^{\frac{0}{\frac{3}{2}\pi}} = 1.$$

(c) $\lim\limits_{x\to -\infty} \frac{8x^8-45x^6+3.3x^2-123}{9x^8+x^7-654x^6+9.81x^4-x^2}$

$\ldots = \lim\limits_{x\to-\infty} \frac{x^8}{x^8} \frac{8-45x^{-2}+3.3x^{-6}-123x^{-8}}{9+x^{-1}-654x^{-2}+9.81x^{-4}-x^{-6}} = \lim\limits_{x\to-\infty} 1 \cdot \lim\limits_{x\to-\infty} \frac{8-45x^{-2}+3.3x^{-6}-123x^{-8}}{9+x^{-1}-654x^{-2}+9.81x^{-4}-x^{-6}} = 1\cdot\frac{8}{9} = \frac{8}{9}.$

(d) $\lim\limits_{x\to 2} f(x)$, where

$$f(x) = \begin{cases} 3e^{\ln\left(\frac{x}{3}\right)} & \text{if } 0 < x \le 2, \\ x^2+7x-16 & \text{if } x > 2. \end{cases}$$

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} 3e^{\ln\left(\frac{x}{3}\right)} = \lim_{x\to 2^-} 3\cdot\frac{x}{3} = \lim_{x\to 2^-} x = 2, \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} x^2+7x-16 = 4+14-16 = 2, \\ \lim_{x\to 2} f(x) &= 2. \end{aligned}$$

4. Find the indicated derivatives. You do not need to simplify. **3 marks each = 9 marks**

   (a) $f(x) = 73.4\sqrt[5]{x^3} - x\cos x$. Find $f'(x)$.

   $f'(x) = 73.4 \cdot \frac{3}{5} \cdot x^{-\frac{2}{5}} - \cos x + x \sin x$.

   (b) $g(x) = \frac{2^x}{x^2+x+1}$. Find $g'(x)$.

   $g'(x) = \frac{\ln 2 \cdot 2^x \cdot (x^2+x+1) - 2^x \cdot (2x+1)}{(x^2+x+1)^2}$.

   (c) $h(x) = (x-1)\ln\left(x^2\right)$. Find $h'(x)$.

   $h'(x) = \ln\left(x^2\right) + 2(x-1) \cdot \frac{1}{x}$.

5. Find the indicated derivatives. You do not need to simplify. **3 marks each = 6 marks**

   (a) $yx = x^2 + 3y^2x^3$. Find $\frac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}(xy) &= \frac{d}{dx}\left(x^2 + 3y^2x^3\right) \Leftrightarrow \\
\frac{dy}{dx} \cdot x + y &= 2x + 6y \cdot \frac{dy}{dx} \cdot x^3 + 9y^2x^2 \Leftrightarrow \\
\frac{dy}{dx}\left(x - 6x^3y\right) &= 2x + 9x^2y^2 - y \Leftrightarrow \\
\frac{dy}{dx} &= \frac{2x + 9x^2y^2 - y}{x - 6x^3y},
\end{aligned}$$

   (b) $f(x) = 3\sin x$. If n is a positive and even integer, find $f^{(n)}(x)$.

$$f'(x) = 3\cos x, \quad f''(x) = -3\sin x, \quad f'''(x) = -3\cos x, \quad f^{(4)}(x) = 3\sin x.$$

   After taking four derivatives we have $f^{(4)} = f$, so we can conclude that

$$f^{(n)}(x) = 3(-1)^{\frac{n}{2}}\sin x.$$

6. Given is the function
$$f(x) = \sqrt{2x+8}.$$

6 marks

(a) Give the equation of the tangent line to the graph of f at $x = 12$.

Find a point on the line. $f(12) = \sqrt{32} = 4\sqrt{2}$ and thus $(12, 4\sqrt{2})$ is a point on the line.
Find the slope of the line by computing the derivative:
$$f'(x) = \frac{1}{\sqrt{2x+8}}, \quad f'(12) = \frac{1}{\sqrt{32}} = \frac{1}{4\sqrt{2}}.$$
Finally use the point-slope formulation to write down the equation of the tangent line:
$$y = \frac{1}{4\sqrt{2}}(x - 12) + 4\sqrt{2}.$$

(b) Give the equation of the tangent line to the graph of f at $x = -4$.

There is a certain ambiguity in this question, hence any of the following two answers is counted as correct:
Answer 1: The tangent line is the line that touches the graph. $f'(-4)$ DNE, more precisely $\lim_{x \to -4^+} f'(x) = \infty$ and thus the tangent line is vertical at $x = -4$. The equation of the tangent line is therefore $x = -4$

Answer 2: The tangent line does not exist, because the tangent line is the line with $f'(-4)$ as slope. Since $x = -4$ is the endpoint of the domain the derivative $f'(-4)$ (as two-sided limit of the difference quotient) is not well defined.

7. Given is the function
$$f(x) = \frac{4(4+3x)}{(x+2)^2}.$$
The first and second derivative are given by
$$f'(x) = \frac{-4(3x+2)}{(x+2)^3}, \qquad f''(x) = \frac{24x}{(x+2)^4}.$$
[15 marks]

(a) What is the domain of f?

$D_f = \mathbb{R} \setminus \{-2\}$,

(b) Compute any x- and y-intercepts of f.

$f(0) = 4$, so $(0,4)$ is the y-intercept.
$$f(x) = 0 \Leftrightarrow 4 + 3x = 0 \Leftrightarrow x = -\frac{4}{3},$$
so $\left(-\frac{4}{3}, 0\right)$ is the x-intercept.

(c) Find all horizontal and vertical asymptotes of f.

The only candidate location for a vertical asymptote is $x = -2$. Compute the following limits to check that there is indeed a vertical asymptote at $x = -2$:
$$\lim_{x \to -2^{\pm}} f(x) = -\infty.$$
To find horizontal asymptotes compute the limits
$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x}{x^2} \frac{4\left(\frac{4}{x} + 3\right)}{\left(1 + \frac{2}{x}\right)^2} = 0 \cdot 12 = 0.$$
Thus there is a horizontal asymptote at $y = 0$.

Question 7 continues on the following page.

(d) Find the intervals on which f is increasing and decreasing. Find any relative extrema of f and identify them as maxima or minima.

$f(x)$ DNE at $x = -2$
$f'(x)$ DNE at $x = -2$, but $-2 \notin D_f$
$f'(x) = 0 \Leftrightarrow 3x + 2 = 0 \Leftrightarrow x = -\frac{2}{3}$

Use the numbers $x = -2$ and $x = -\frac{2}{3}$ to divide the real number line into three intervals: $(-\infty, -2), \left(-2, -\frac{2}{3}\right)$, and $\left(-\frac{2}{3}, \infty\right)$. Compute the sign of f' on each interval: -, +, - respectively. Therefore f is decreasing on $(-\infty, -2)$, increasing on $\left(-2, -\frac{2}{3}\right)$, and decreasing on $\left(-\frac{2}{3}, \infty\right)$.
Furthermore we can now conclude that there is a local maximum for $x = -\frac{2}{3}$ with value $f\left(-\frac{2}{3}\right) = \frac{9}{2}$.

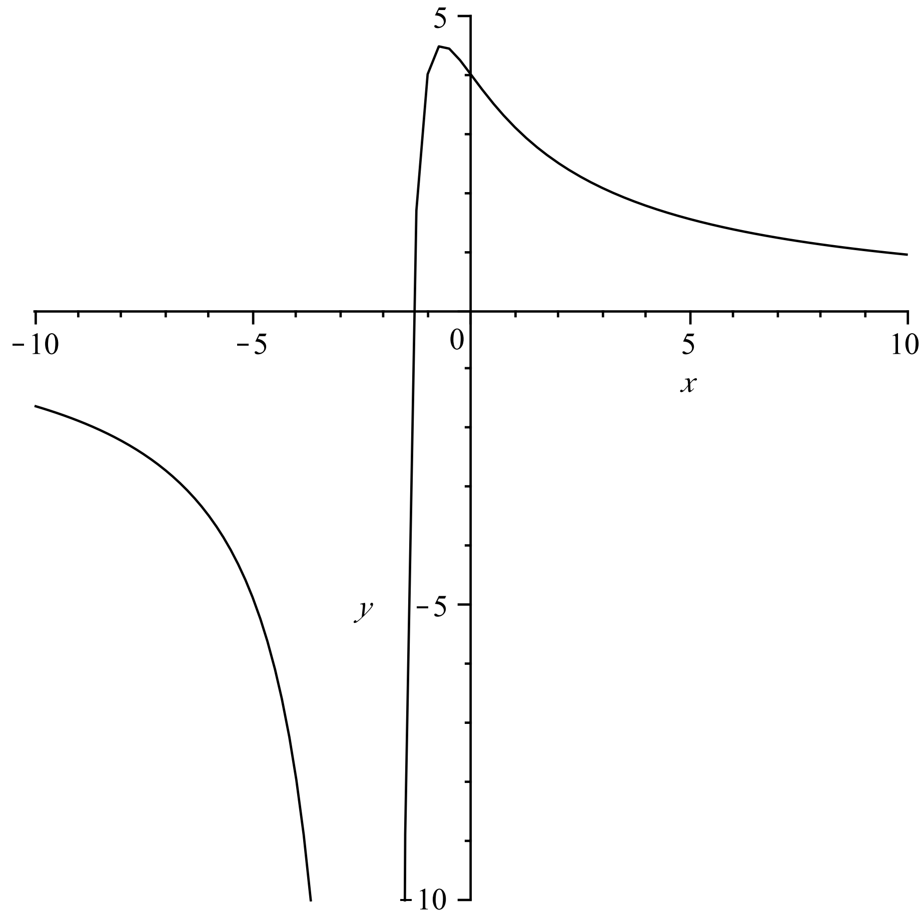
(e) Find all intervals where f is concave up or concave down and all inflection points.

$f(x)$ DNE at $x = -2$
$f''(x)$ DNE at $x = -2$, but $-2 \notin D_f$
$f''(x) = 0 \Leftrightarrow x = 0$

Use the numbers $x = -2$ and $x = 0$ to divide the real number line into three intervals: $(-\infty, -2), (-2, 0)$, and $(0, \infty)$. Compute the sign of f'' on each interval: -, -, + respectively. Therefore f is concave down on $(-\infty, -2)$ and $(-2, 0)$ and concave up on $(0, \infty)$.
Furthermore we can now conclude that there is an inflection point in$x = 0$. $f(0) = 4$, so $(0, 4)$ is an inflection point.

Question 7 continues on the following page.

(f) Use the information you have found to sketch the graph of f in the coordinate system below.



8. A pencil producing company has researched the demand of its customers and found that the demand q, given in units of 100,000 pencils per year, depends on the price p per pencil in dollars in the following way:

$$q = f(p) = 24p - 16p^2, \qquad p \in \left(0, \frac{3}{2}\right).$$

8 marks

(a) Give an expression $E(p)$ for the elasticity of demand in terms of the price p and determine for which price intervals the demand is elastic and for which it is inelastic.

Elasticity:

$$E(p) = -\frac{p}{q(p)}\frac{dq(p)}{dp} = \frac{-p}{24p-16p^2}(24-32p) = \frac{32p-24}{24-16p}.$$

Compute

$$E(p) = 1 \Leftrightarrow 32p - 24 = 24 - 16p \Leftrightarrow p = 1.$$

Since $E(p)$ is increasing as function of p, we find that $E(p) < 1$ (inelastic) if $p < 1$ and $E(p) > 1$ (elastic) if $p > 1$.

(b) Determine the price at which the demand is at its maximum and the maximum value of the demand.

Find the critical numbers of f:
$f'(p)$ DNE: no such p,
$f'(p) = 0 \Leftrightarrow 24 - 32p = 0 \Leftrightarrow p = \frac{3}{4}$.
So $p = \frac{3}{4}$ is the only critical number of f.
Compute $f\left(\frac{3}{4}\right) = 9$. To check the value near the endpoints of the domain, we compute

$$\lim_{p\to 0^+} f(p) = 0, \qquad \lim_{p\to \frac{3}{2}^-} f(p) = 36 - 36 = 0.$$

Comparing these three values, we find that the maximum is taken at $p = \frac{3}{4}$ and its value is 9. We conclude that the demand is maximum at a price of \$0.75 and the demand is then $900,000$ pencils per year.

(c) The company can satisfy every demand, i.e. it sells exactly the number of pencils that is demanded, no matter how high the demand is. For which price should the pencils be sold so that the

revenue is at a maximum? What is the revenue *per year*[1] in that case?

The revenue is at a maximum when $E(p) = 1$, so when $p = 1$. $f(1) = 24 - 16 = 8$, so the demand when the price is \$1 is 800,000 units per year. Therefore the maximum revenue per year is $\$800,000$.

This can also be computed by finding the maximum directly from the function $R(p) = p \cdot q = p \cdot f(p) = 24p^2 - 16p^3$.

[1] Missing in the printed exam, announced during the exam.

9. Given is the function

$$f(x) = \frac{1}{3}x^3 + 1.75x^2 - 15x + 1$$

and its derivative

$$f'(x) = x^2 + 3.5x - 15.$$

The function f has a root between $x = 2$ and $x = 5$. We are going to use Newton's method to approximate this root. **7 marks**

(a) We start by making an initial guess. Explain which of the following three choices is the best initial guess: $c_1 = -6$, $c_1 = 2.5$, or $c_1 = 4.2$.

The first iteration step in Newton's method is

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)}.$$

We compute $f'(-6) = f'(2.5) = 0$ and $f(4.2) = 17.34 \neq 0$. The values $c_1 = -6$ and $c_1 = 2.5$ therefore cannot be used as initial value and $c_1 = 4.2$ is the best choice.
Alternatively, the choice $c_1 = -6$ might also be rejected because the value -6 is the furthest removed from the interval of interest.

(b) Starting from the initial guess you chose in 9a compute three further approximations c_2, c_3, c_4 of the root, using Newton's method.

The general formula to compute c_{n+1} from c_n is

$$c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)}.$$

Using this with $c_1 = 4.2$ we find

$$\begin{aligned} c_2 &= 4.2 + \frac{6.434}{17.34} \approx 4.57105, \\ c_3 &= 4.57105 - \frac{0.83622}{21.89317} \approx 4.53285, \\ c_4 &= 4.5385 - \frac{0.00911}{21.41170} \approx 4.53242. \end{aligned}$$

(c) Based on your answers in 9b give an approximation of the root we are looking for, with two decimals accuracy.

4.53

10. Jack wants to be able to buy a new car for \$50,000 in five years time. If he invests today then he will get a yearly interest rate of 3%, which is compounded every half year. **9 marks**

   (a) Give an expression for $A(P)$, the compound amount in dollars after five years, if Jack invests P dollars today. Hint: Have a look back at question 1 of this exam.

   $r = 0.03$ is the yearly interest rate, $m = 2$ is the number of compounding periods per year and $t = 5$ is the total number of years. We thus have

$$A(P) = P\left(1 + \frac{r}{m}\right)^{tm} = P\left(1 + \frac{0.03}{2}\right)^{2\cdot 5} = P(1.015)^{10}.$$

   (b) How much does Jack need to invest today to have \$50,000 in five years time? Give your answer accurate to the nearest cent.

   $A = 50,000$, so $P = \frac{50,000}{(1.015)^{10}} \approx 43083.36$. Jack needs to invest \$43083.36 today.

   (c) Another option Jack considers, is to set up an annuity *in an account that pays an interest rate of 3% per year, compounded annually*[2]. He computes that if he pays R dollars at the end of each year, with the first payment being a year from now, then the amount of the annuity after five years as a function of R is $S = R\left(\frac{(1.03)^5-1}{0.03}\right)$ dollars. What amount does Jack need to pay per year for the annuity to have a value of \$50,000 after five years? Give your answer accurate to the nearest cent.

   $S = 50,000$, so $R = 50,000 \cdot \left(\frac{(1.03)^5-1}{0.03}\right)^{-1} \approx 9417.73$. Jack needs to pay \$9417.73 per year.

Question 10 continues on the following page.

[2]Missing in the printed exam, announced during the exam.

(d) What is the present value of the annuity if Jack yearly pays the amount you found in 10c? Give your answer accurate to the nearest cent. Hint: Have another look at question 1 of this exam.

The number of compounding periods per year is $m = 1$, the interest rate per period is $i = \frac{0.03}{1} = 0.03$, the total number of periods is $n = 5$ and the rent per year is $R = 9417.73$. The present value of the annuity is then

$$P_{\text{annuity}} = R\left(\frac{1-(1+i)^{-n}}{i}\right) = 9417.73\left(\frac{1-(1.03)^{-5}}{0.03}\right) \approx 43130.45.$$

The present value of the annuity is \$43130.45.

(e) Explain which is the cheaper choice for Jack, an investment today or yearly payments for the annuity?

Comparing the answers of (b) and (d) we see that the first option, i.e. an investment today, has the lowest present value and is thus the cheaper choice.

Another interpretation leads to a comparison between the payment for the investment (answer of (b)) and the 5 times the yearly payment in (c). Since $5 \cdot \$9417.73 = \$47088.65$, this also leads to the conclusion that an investment today is cheaper than the annuity.

11. Given is the function

$$f(x) = \sin(12\pi x).$$

6 marks

(a) Use differentials (i.e. linear approximation) to approximate the value of $f\left(\frac{3}{4} + \Delta x\right)$, for the following values of Δx:

i. $\Delta x = 0.1$

ii. $\Delta x = 0.4$

iii. $\Delta x = 0.05$

Give your answer accurate up to three decimals.

We compute $f\left(\frac{3}{4}\right) = \sin(9\pi) = 0$, $f'(x) = 12\pi\cos(12\pi x)$, and $f'\left(\frac{3}{4}\right) = 12\pi\cos(9\pi) = -12\pi$. Thus the linear approximation is given by:

$$f\left(\frac{3}{4} + \Delta x\right) \approx L(\Delta x) = f\left(\frac{3}{4}\right) + f'\left(\frac{3}{4}\right) \cdot \Delta x = -12\pi\Delta x.$$

Computation now gives

i. $f\left(\frac{3}{4} + 0.1\right) \approx -3.770$

ii. $f\left(\frac{3}{4} + 0.4\right) \approx -15.080$

iii. $f\left(\frac{3}{4} + 0.05\right) \approx -1.885$

(b) Explain for which of the cases in 11a the linear approximation via differentials is not a good way to approximate the value of $f(\frac{3}{4} + \Delta x)$.

The function f takes values in $[-1, 1]$, so none of the cases in (a) is accurate.

Another, more intricate, argument is that one cannot expect a linear approximation for $\sin(12\pi x)$ around $x = \frac{3}{4}$ to be accurate over a distance more than a quarter period (look at the graph to see why). The period here is $\frac{2\pi}{12\pi} = \frac{1}{6}$. All of the Δx in (a) are larger than $\frac{1}{24} \approx 0.042$, i.e. larger than one quarter period.

12. A manufacturer of handcrafted wine racks has determined that the cost in dollars to produce x units per month is given by

$$C = 0.3x^2 + 11,000.$$

6 marks

(a) How fast is the cost per month changing when production is increasing at a rate of 10 units per month and the production level is 75 units?

If we consider both C and x to be functions of time t, we have a related rates problem:

$$\frac{dC}{dt} = 0.6 \cdot x \cdot \frac{dx}{dt}.$$

We have $\frac{dx}{dt} = 10$ and $x = 75$. Therefore

$$\frac{dC}{dt} = 0.6 \cdot 75 \cdot 10 = 450$$

and thus the cost per month is increasing at a rate of \$450 per month.

(b) The manufacturer has also found a relation between the cost C and revenue R (both in dollars) in one month:

$$C = \frac{R^2}{200,000} + 17500.$$

Find the rate of change of revenue with respect to time when the monthly revenue is \$25,000 and the cost is changing at the rate you found in question 12a.

Consider both C and R as functions of time t. Then:

$$\frac{dC}{dt} = \frac{R}{100,000}\frac{dR}{dt}, \quad \text{so} \quad \frac{dR}{dt} = \frac{100,000}{R}\frac{dC}{dt}.$$

We have $\frac{dC}{dt} = 450$ and $R = 25,000$. Therefore

$$\frac{dR}{dt} = \frac{100,000}{25,000} \cdot 450 = 4 \cdot 450 = 1800$$

and thus the monthly revenue is increasing at a rate of \$1800 per month.