 Version #2

(1) [Marks: 4] The amount of water a city uses when the population is p is $W(p)$ litres. The population of this city in year t is $P(t)$. Interpret the following functions, or explain why it doesn't make sense;

(a) $P \circ W$ $W(p)$ is litres P requires time (years) as input, so $(P \circ W)(p) = P(W(p))$ does not make sense

(b) $W \circ P$ $P(t)$ is population. W requires population as input, so $(W \circ P)(t) = W(P(t))$ does make sense;
$(W \circ P)(t)$ is amount of water city uses in year t.

(2) [Marks: 6] Determine the following limits as either a number, $\pm\infty$, or DNE (does not exist). If DNE explain why the limit does not exist. (**Read the limits carefully**.)

(a) $\lim_{x \to 3^-} \frac{2(x-3)}{\sqrt{6-2x}} \times \frac{\sqrt{6-2x}}{\sqrt{6-2x}}$

$= \lim_{x \to 3^-} \frac{2(x-3)\sqrt{6-2x}}{6-2x}$

$= \lim_{x \to 3^-} -\sqrt{6-2x} = 0$

(b) $\lim_{x \to -2} \dfrac{4x+8}{-3|x+2|}$

$$\lim_{x \to -2^-} \frac{4x+8}{-3|x+2|} = \lim_{x \to -2^-} \frac{4x+8}{-3(-x-2)} = \frac{4}{3}$$

$$\lim_{x \to -2^+} \frac{4x+8}{-3|x+2|} = \lim_{x \to -2^+} \frac{4x+8}{-3(x+2)} = -\frac{4}{3}$$

one-sided limits not equal

$\rightarrow$ limit does not exist

(c) $\lim_{x \to \infty} \dfrac{2x^5 - 3x + 5}{3x^4 + 100x^3 + 75}$

$$= \lim_{x \to \infty} \frac{x^4\left(2x - \frac{3}{x^3} + \frac{5}{x^4}\right)}{x^4\left(3 + \frac{100}{x} + \frac{75}{x^4}\right)}$$

$$= \lim_{x \to \infty} \frac{2x - \frac{3}{x^2} + \frac{5}{x^4}}{3 + \frac{100}{x} + \frac{75}{x^4}} \quad \begin{array}{l} \leftarrow \lim_{x \to \infty} = \infty \\ \leftarrow \lim_{x \to \infty} = 3 \end{array} \Bigg\} \text{ limit} = \infty$$

(does not exist)

(3) [Marks: 6] The percentage of grade 12 students that will enter the psychology program at the university was 2.5% at the beginning of 2001 ($x = 0$) and is projected to grow linearly so that by the beginning of 2009 the percentage entering psychology is projected to be 3.5%.

(a) Find an equation which expresses the relationship between the percentage of students entering psychology (p) and the number of years from 2001 (x).

slope $m = \dfrac{3.5 - 2.5}{8} = \dfrac{1}{8}$ % per year

$p(0) = 2.5 = b$

$\rightarrow \quad p(x) = \frac{1}{8}x + 2.5$

(b) During what year will the percentage of students entering psychology reach 12.5%?

Find x when $p(x) = 12.5$;

$12.5 = \frac{1}{8}x + 2.5$

$\rightarrow \quad 10 = \frac{1}{8}x \quad \rightarrow x = 80 \quad \rightarrow$ year $\underline{2081}$

(c) What is the projected percentage of students entering psychology in the year 2061?

$x = 60$

$P(60) = \frac{1}{8}(60) + 2.5$

$= 7.5 + 2.5 = 10\ \%$

(4) [Marks: 8] **(a)** Find a point on the graph of $y = \dfrac{3}{x^2} + 2$ where the tangent line has slope 3/4 and find the equation of this tangent line (in point-slope form; $y = mx + b$). Use the Four-step process to compute the derivative (i.e., the definition of the derivative) in this problem (if you cannot find the derivative this way you can use the differentiation rules but you will not get any marks for finding the derivative this way).

$$y'(x) = \lim_{h \to 0} \frac{\frac{3}{(x+h)^2} + 2 - \left(\frac{3}{x^2} + 2\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3x^2 - 3(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h \to 0} \frac{-6xh - 3h^2}{hx^2(x+h)^2} = -\frac{6}{x^3}$$

Solve $y'(x) = \frac{3}{4}$ for x:

$$\frac{3}{4} = -\frac{6}{x^3} \rightarrow x^3 = -8 \rightarrow x = -2 \rightarrow y(-2) = \frac{11}{4}$$

point on graph: $\left(-2, \frac{11}{4}\right)$. Equation of line;

$$m = \frac{3}{4} = \frac{y - \frac{11}{4}}{x+2} \rightarrow \boxed{y = \frac{3}{4}x + \frac{17}{4}}$$

(b) Suppose this function represents the cost C (in dollars) to produce x units of a commodity, that is $C(x) = \dfrac{3}{x^2} + 2$.

(bi) What is the marginal cost of this commodity?

$$C'(x) = -\frac{6}{x^3}$$

(bii) Use marginal cost to estimate the cost difference between producing 5 and 6 units of this commodity.

$$C(6) - C(5) \approx C'(5) = -\frac{6}{125}$$

(5) [Marks: 6] Use the rules of differentiation to find the derivatives of the following functions. You do not need to simplify your answer.

(a) $h(x) = 2x^{-2/3} + 5x^4 - 2x + 1$

$$h' = 2\left(-\tfrac{2}{3}\right)x^{-5/3} + 20x^3 - 2$$

$$= -\frac{4}{3}x^{-5/3} + 20x^3 - 2$$

(b) $k(t) = (2t^3 + 7)^{4/5}$

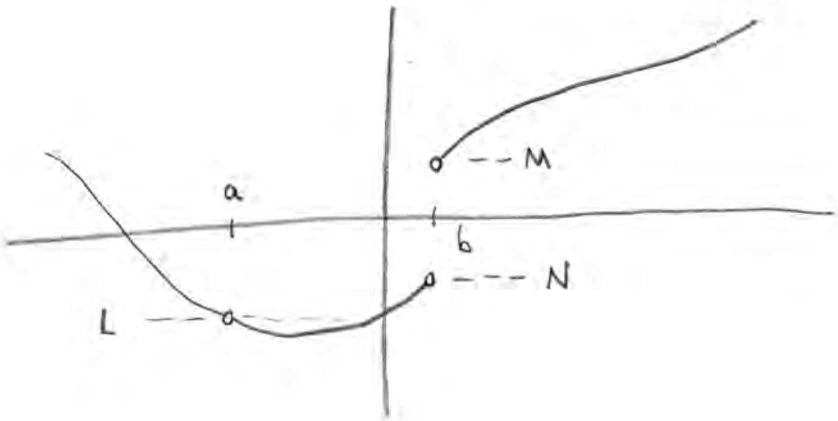
$$k'(t) = \frac{4}{5}\left(2t^3+7\right)^{-1/5} \cdot \left(6t^2\right)$$

$$= \frac{24}{5}t^2\left(2t^3+7\right)^{-1/5} = \frac{24t^2}{5\sqrt[5]{2t^3+7}}$$

(c) $f(w) = \dfrac{2w + 4w^3 - 3}{4w^2 + 3w}$

$$f'(w) = \frac{(2+12w^2)(4w^2+3w) - (2w+4w^3-3)(8w+3)}{(4w^2+3w)^2}$$

(6) [Marks: 4] (a) Give an example of a function that has two different types of discontinuity, one at $x = a$ and one at $x = b$. Label the types of discontinuity and explain why the function is discontinuous there (for example, you will have to discuss certain limits). Your may answer may include a sketch of such a function and/or a formula for the function.



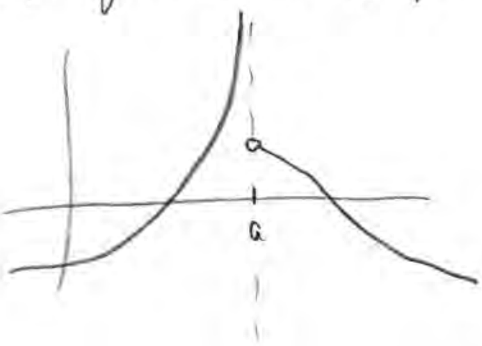
at $x=a$ there is a removable discontinuity

$\lim_{x \to a} f(x) = L$ (exists), but $\neq f(a)$ (since $f(a)$ not defined)

at $x=b$ there is a jump discontinuity

$\lim_{x \to b^-} f(x) = N$, $\lim_{x \to b^+} f(x) = M$ and $M \neq N$.

Another type of discontinuity:



here, $\lim_{x \to a^-} f(x) = \infty$ (DNE)

So $\lim_{x \to a} f(x)$ does not exist