MATH 155
MIDTERM 2

8:30-9:20, March 10, 2004
Instructor: Peter Berg

Family name: ______________ **Initials:** _______

Student ID number: ______________

READ INSTRUCTIONS CAREFULLY:

- **Do not lift the cover page until instructed!**
- Fill out your name and ID in the space provided.
- You MUST NOT use a calculator. NO other aids.
- Answer all questions, explaining your answers carefully in the space provided. If you run out of space, use the back of the preceding page.
- This exam consists of 5 questions and 6 pages (including this one).

Question	1	2	3	4	5	Total
Grade	/10	/10	/10	/10	/10	/50

Good Luck!

1. (10 marks) Use partial fractions to solve

$$\int_0^1 \frac{dx}{2x^2 - 8x - 10}.$$

Solution: First we write the integrand as

$$\frac{1}{2x^2 - 8x - 10} = \frac{1}{2(x+1)(x-5)}.$$

Then we use partial fractions to write the integrand as

$$\frac{1}{2x^2 - 8x - 10} = \frac{1}{12}\left(\frac{1}{x-5} - \frac{1}{x+1}\right).$$

Therefore, the integral becomes

$$\begin{aligned}
\int_0^1 \frac{dx}{2x^2 - 8x - 10} &= \frac{1}{12}\int_0^1 \left(\frac{1}{x-5} - \frac{1}{x+1}\right) dx \\
&= \frac{1}{12}\left(\ln|x-5| - \ln|x+1|\right)\Big|_0^1 \\
&= \frac{1}{12}\left(\ln\left|\frac{-4}{-5}\right| - \ln\left|\frac{2}{1}\right|\right) \\
&= \frac{1}{12}\ln\left(\frac{2}{5}\right).
\end{aligned}$$

2. (10 marks) Determine the constant c so that

$$\int_0^\infty \frac{c\,e^{-2x}}{1+e^{-2x}}\,dx = 1\,.$$

Solution: We evaluate the integral

$$\begin{aligned}
\int_0^\infty \frac{c\,e^{-2x}}{1+e^{-2x}}\,dx &= -\frac{c}{2}\int_0^\infty \frac{-2\,e^{-2x}}{1+e^{-2x}}\,dx \\
(z = e^{-2x}) &= -\frac{c}{2}\int_1^0 \frac{1}{1+z}\,dz \\
&= -\frac{c}{2}\ln|1+z|\Big|_1^0 \\
&= \frac{c}{2}\ln 2 \\
&= 1\,. \\
&\Rightarrow\ c = \frac{2}{\ln 2}
\end{aligned}$$

3. (10 marks) Determine the Taylor polynomial of degree 5, $P_5(x)$, of the function $f(x) = e^{-2x}$ about $x = 0$. Compare its value at $x = 1$ to $f(1) = e^{-2} \approx \frac{2}{15}$.

Solution: We find

$$
\begin{aligned}
f(0) &= 1 \\
f'(0) &= -2 \\
f''(0) &= 4 \\
f^{(3)}(0) &= -8 \\
f^{(4)}(0) &= 16 \\
f^{(5)}(0) &= -32
\end{aligned}
$$

and therefore

$$
\begin{aligned}
P_5(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4 + \frac{f^{(5)}(0)}{120}x^5 \\
&= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5 \,.
\end{aligned}
$$

We calculate

$$f(1) = \frac{1}{e^2} \approx \frac{1}{7.28} \approx \frac{2}{15}$$

and

$$P_5(1) = 1 - 2 + 2 - \frac{4}{3} + \frac{2}{3} - \frac{4}{15} = \frac{1}{15}\,.$$

The approximation is rather poor and differs by a factor of $\frac{1}{2}$.

4. (10 marks) Find the solution $y(x)$ of

$$\frac{dy}{dx} = y(2-y)$$

with $y(0) = 1$. What is $\lim_{x\to\infty} y(x)$?

Solution: We separate the variables

$$\int_1^y \frac{dz}{z(2-z)} = \int_0^x dw = x\,.$$

We use partial fractions to write for the integrand

$$\frac{1}{z(2-z)} = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{2-z}\right)\,.$$

Hence,

$$\begin{aligned}
\frac{1}{2}\int_1^y \left(\frac{1}{z} + \frac{1}{2-z}\right) dz &= x \\
\Rightarrow \frac{1}{2}\left(\ln|z| - \ln|2-z|\right)\Big|_1^y &= x \\
\Rightarrow \frac{1}{2}\left(\ln|y| - \ln|2-y|\right) &= x \\
\Rightarrow \frac{1}{2}\ln\left|\frac{y}{2-y}\right| &= x \\
\Rightarrow \left|\frac{y}{2-y}\right| &= e^{2x} \\
(y(0)=1) \Rightarrow \frac{y}{2-y} &= e^{2x} \\
\Rightarrow y(x) &= \frac{2e^{2x}}{1+e^{2x}}\,.
\end{aligned}$$

Check: $y(0) = 1$ is fulfilled.

We find

$$\lim_{x\to\infty} y(x) = 2\,.$$

5. (10 marks) Determine the equilibria of

$$\frac{dy}{dx} = y(2-y)\,.$$

Analyze the local stability of the equilibria analytically. Is one of them globally stable on $(0,2]$ (graphical argument)?

We define

$$\frac{dy}{dx} = y(2-y) =: g(y).$$

Then the equilibria $\hat{y}$ are determined by

$$g(y) = 0 \ \Rightarrow\ \hat{y}_1 = 0 \ \text{ and } \ \hat{y}_2 = 2\,.$$

The local stability is determined by $g'(\hat{y}) = 2 - 2\hat{y}$:

$$g'(\hat{y}_1) = g'(0) = 2 > 0 \ \Rightarrow\ \hat{y}_1 \text{ is locally unstable}$$
$$g'(\hat{y}_2) = g'(2) = -2 < 0 \ \Rightarrow\ \hat{y}_2 \text{ is locally stable}$$

From the figure below we see that $\hat{y}_2 = 2$ is globally stable on $(0,2]$ since $g(y) > 0$ on $(0,2)$. Therefore $\frac{dy}{dx} > 0$ on $(0,2)$, and y grows and approaches $y = 2$.