GREEN AND BLUE PAPERS

SIMON FRASER UNIVERSITY

MATH 155 Midterm 2

5 March 2008, 08:30–09:20

Last Name ____________________

Given Name(s) ____________________

Student # ____________________

Signature ____________________

INSTRUCTIONS

1. **Do not open this booklet until told to do so.**
2. Write your last name, given name(s), and student number in the box above. Sign on the last line of the box.
3. This exam has 6 questions on 6 pages. Check to make sure that your exam is complete.
4. No book, paper or device other than usual writing instruments, this examination booklet, and a scientific calculator are allowed. **Calculators with graphing and/or symbolic computation capabilities are not allowed.**
5. **During the examination, speaking to, communicating with, or exposing written papers to the view of other examinees is forbidden.**
6. You may use the **reverse side of the previous page** for rough work or if you run out of space.
7. **You may lose marks if your explanations are incomplete or poorly presented.**
8. **Stop writing when you are instructed to do so. Failure to follow instructions may result in penalties.**

Question	Maximum	Score
1	8	
2	8	
3	8	
4	6	
5	8	
6	7	
Total	45	

[8] 1. Evaluate $\int_2^5 \frac{x+7}{x^2-x-6}\,dx$.

$$x^2-x-6=0 \qquad x_{1,2}=\frac{1\pm\sqrt{1+24}}{2}=\left\langle\begin{matrix}3\\-2\end{matrix}\right.$$

$$x^2-x-6=(x-3)(x+2)$$

$$\frac{x+7}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}=$$

$$=\frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}$$

$$A+B=1 \Rightarrow B=1-A$$
$$2A-3B=7$$
$$2A-3(1-A)=7 \quad \Downarrow$$
$$5A=10$$
$$A=2 \Rightarrow B=-1$$

discontinuity at $x=3$

$$\int_2^5\frac{x+7}{x^2-x-6}\,dx=\int_2^3\frac{x+7}{x^2-x-6}\,dx+\int_3^5\frac{x+7}{x^2-x-6}\,dx=$$

$$=\int_2^3\left(\frac{2}{x-3}+\frac{-1}{x+2}\right)dx+\int_3^5\left(\frac{2}{x-3}+\frac{-1}{x+2}\right)dx$$

$$\int_2^3\frac{2}{x-3}\,dx=\lim_{z\to3^-}\int_2^z\frac{2\,dx}{x-3}=\lim_{z\to3^-}\Big[2\ln|x-3|\Big]_2^z=$$

$$=\lim_{z\to3^-}2\ln(3-z)=-\infty$$

Therefore $\int_2^5\frac{x+7}{x^2-x-6}\,dx$ <u>diverges.</u>

[8] 2. Evaluate $\int_0^\infty \frac{dx}{x^2+9}$.

$$\int_0^\infty \frac{dx}{x^2+9} = \lim_{z\to\infty} \int_0^z \frac{dx}{x^2+9}$$

$$\boxed{\begin{array}{c} u = \frac{x}{3} \\ 3du = dx \end{array}}$$

$$\int_0^z \frac{dx}{x^2+9} = \int_0^z \frac{1}{9}\cdot\frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx =$$

$$= \int_0^{z/3} \frac{1}{3}\cdot\frac{1}{u^2+1}\,du = \frac{1}{3}\Big[\tan^{-1} u\Big]_0^{z/3} =$$

$$= \frac{1}{3}\tan^{-1}\frac{z}{3}$$

$$\int_0^\infty \frac{dx}{x^2+9} = \lim_{z\to\infty} \frac{1}{3}\tan^{-1}\frac{z}{3} = \frac{1}{3}\cdot\frac{\pi}{2} = \frac{\pi}{6}$$

[8] **3.** Evaluate $\int x(\ln x)^2\,dx$.

$$\boxed{u=(\ln x)^2,\quad v'=x,\quad v=\frac{x^2}{2}}$$

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \int (2\ln x)\cdot\frac{1}{x}\cdot\frac{x^2}{2}\,dx$$

$$=\frac{x^2}{2}(\ln x)^2 - \int x\ln x\,dx =$$

$$\boxed{\begin{array}{l} u=\ln x \\ v'=x, \\ v=\frac{x^2}{2}\end{array}}$$

$$=\frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx\right) =$$

$$=\frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$$

[6] **4.** Use the midpoint rule with $n = 5$ intervals to find an approximation for $\int_2^3 \frac{1}{x^2}\,dx$. Compare the approximation with the exact value of the integral.

$$\Delta x = \frac{3-2}{5} = 0.2$$

$x_0 = 2 \quad x_1 = 2.2 \quad x_2 = 2.4 \quad x_3 = 2.6 \quad x_4 = 2.8 \quad x_5 = 3$

$c_1 = 2.1 \quad c_2 = 2.3 \quad c_3 = 2.5 \quad c_4 = 2.7 \quad c_5 = 2.9$

$$\int_2^3 \frac{dx}{x^2} \approx 0.2\left(\frac{1}{2.1^2} + \frac{1}{2.3^2} + \frac{1}{2.5^2} + \frac{1}{2.7^2} + \frac{1}{2.9^2}\right) =$$

$$= 0.166374712\ldots$$

The exact value is

$$\int_2^3 \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_2^3 = -\frac{1}{3} - \left(-\frac{1}{2}\right) = \frac{1}{6}.$$

[6] 5. (a) Compute the Taylor polynomial of degree 4 about $x = 0$ for the function $f(x) = \ln(1+x)$.

$f(x) = \ln(1+x)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\quad f'''(0) = 2$

$f^{(4)}(x) = -\frac{6}{(1+x)^4}$ $\quad f^{(4)}(0) = -6$

$$P_4(x) = 0 + 1\cdot x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 =$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

[2] (b) Use the polynomial found in part (a) to approximate the value $\ln(0.8)$. Compare this approximation with the exact value of $\ln(0.8)$.

$\ln(0.8) \approx P_4(-0.2) = -0.22306666\ldots$

The exact value is

$\ln(0.8) = -0.22314355\ldots$

[7] 6. Let $T(t)$ denote the temperature of a small stone at time t. At time $t = 0$ the stone is submerged in a large tank of water whose temperature is constant and denoted by M. Suppose that the rate of change of T is determined by

$$\frac{dT}{dt} = 0.4(M - T).$$

Assume that $T(0) = 30$ and $M = 20$. Find $T(3)$, i.e. the temperature of the stone at time $t = 3$.

$$\frac{dT}{dt} = 0.4(20-T)$$

$$\frac{dT}{20-T} = 0.4\,dt$$

$$\int \frac{dT}{20-T} = \int 0.4\,dt$$

$$-\ln|20-T| = 0.4t + C_1$$

$$\ln|20-T| = -0.4t + C_2$$

$$|20-T| = e^{-0.4t+C_2}$$

$$20-T = \pm e^{-0.4t+C_2}$$

$$20-T = C_3 \cdot e^{-0.4t}$$

$$T(t) = 20 + C \cdot e^{-0.4t}$$

Plug in the initial condition:

$$30 = T(0) = 20 + C \cdot e^0 \Rightarrow C = 10$$

$$T(t) = 20 + 10 \cdot e^{-0.4t}$$

$$T(3) = 20 + 10e^{-1.2} = 23.01\ldots$$