MATH 155
MIDTERM 1

8:30-9:20, February 11, 2004
Instructor: Peter Berg

Family name: ____________ **Initials:** ______

Student ID number: ____________

READ INSTRUCTIONS CAREFULLY:

- **Do not lift the cover page until instructed!**
- Fill out your name and ID in the space provided.
- You MUST NOT use a calculator. NO other aids.
- Answer all questions, explaining your answers carefully in the space provided. If you run out of space, use the back of the preceding page.
- This exam consists of 4 questions and 5 pages (including this one).

Question	1	2	3	4	Total
Grade	/10	/20	/10	/10	/50

Good Luck!

1. “Sometimes things are simpler than they seem.”

(10 marks) Use the algebraic rules for sums to evaluate

$$\sum_{k=1}^{20}\left[(k+1)(k+2)-(k+1)(k-1)\right].$$

Recall that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \quad (*)$$

<u>Solution:</u>

First we simplify the bracket to find

$$\sum_{k=1}^{20}\left[(k+1)(k+2)-(k+1)(k-1)\right] = \sum_{k=1}^{20} 3k+3.$$

We use summation rules and write

$$\sum_{k=1}^{20} 3k+3 = 3\sum_{k=1}^{20} k + \sum_{k=1}^{20} 3.$$

Now we use (*) and find

$$\begin{aligned} 3\sum_{k=1}^{20} k + \sum_{k=1}^{20} 3 &= 3\frac{20\times 21}{2} + 3\times 20 \\ &= 630+60 \\ &= 690. \end{aligned}$$

2. "Many roads lead to Rome."

a) (10 marks) Solve

$$\int_1^e \frac{\ln x}{x}\,dx$$

using the substitution rule.

<u>Solution:</u>

We set $z = \ln x$ and find

$$dz = dx/x$$

and the limits change like

$$\begin{aligned} x = 1 &\;\Rightarrow\; z = 0 \\ x = e &\;\Rightarrow\; z = 1. \end{aligned}$$

The integral then changes to

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 z\,dz$$

and so

$$\begin{aligned} \int_1^e \frac{\ln x}{x}\,dx &= \int_0^1 z\,dz \\ &= \left.\frac{1}{2}z^2\right|_0^1 \\ &= \frac{1}{2}. \end{aligned}$$

b) (10 marks) Solve

$$\int_1^e \frac{\ln x}{x}\, dx$$

using integration by parts.

<u>Solution:</u>
We set

$$\begin{aligned} v' &= 1/x \\ \Rightarrow v &= \ln x \\ w &= \ln x \\ \Rightarrow w' &= 1/x. \end{aligned}$$

With these definitions we find

$$\int_1^e \frac{\ln x}{x}\, dx \;=\; (\ln x)^2\Big|_1^e - \int_1^e \frac{\ln x}{x}\, dx.$$

Bringing the integral on the right hand side over to the left hand side and division by a factor 2 yields

$$\begin{aligned} \int_1^e \frac{\ln x}{x}\, dx &= \frac{1}{2}\,(\ln x)^2\Big|_1^e \\ &= \frac{1}{2}. \end{aligned}$$

3. This semester's most important theorem(s).

a) (5 marks) State the Fundamental Theorem of Calculus, Part I.

Let $f(x)$ be a <u>continuous</u> function on $[a, b]$ and the function $F(x)$ be defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then $F(x)$ is differentiable on $[a, b)$ with

$$F'(x) = f(x).$$

b) (5 marks) State the Fundamental Theorem of Calculus, Part II.

If $f(x)$ is a <u>continuous</u> function on $[a, b]$ and $G(x)$ is an antiderivative of $f(x)$, i.e.

$$G'(x) = f(x),$$

then

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

4. “An ellipse does not have edges but it has more edges than a circle.” (famous physicist of the 20th century)

   (10 marks) Let us consider the ellipse

$$\frac{x^2}{a^2} + y^2 = 1,$$

   where $a > 0$ is a real number. Determine the volume of the solid obtained by rotating the ellipse about the x-axis, considering the upper branch $y = \sqrt{1 - \frac{x^2}{a^2}}$.

   Solution:

   We would like to use the disk method. The upper and lower limits of integration are a and $-a$, respectively. The volume V is then found to be

$$\begin{aligned} V &= \pi \int_{-a}^{a} y^2 \, dx \\ &= \pi \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx \\ &= \pi \left(x - \frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a} \\ &= 2\pi \left(x - \frac{x^3}{3a^2}\right)\Bigg|_{0}^{a} \\ &= \frac{4}{3}\pi a. \end{aligned}$$

   This makes sense since it is a times the volume of the corresponding sphere of radius 1. Each disk element is stretched by a factor a.