MATH 155 Final Exam

Last Name:________________________________

Given Name(s):____________________________

Student ID #:_____________________________

Signature:__________________________________

INSTRUCTIONS:

1. Print your name and ID # in the spaces given.
2. Sign your name at the indicated place.
3. This exam has 10 questions on Page 2 to Page 11.
3. The allocated time for this exam is 3 hours.
4. Calculators are not allowed.
5. A formula sheet is attached.
6. Answers to all questions must be properly justified.

Questions	Q1	Q2	Q3	Q4	Q5	
Your Scores	/10	/10	/10	/10	/10	
Questions	Q6	Q7	Q8	Q9	Q10	Total
Your Scores	/10	/10	/10	/10	/10	/100

Q1. Evaluate the following expressions.

(a) (2 marks) $\frac{d}{dx}\int e^{x^2}\,dx$

(b) (2 marks) $\int\left(\frac{d}{dx}\sqrt{1+x^{2004}}\right)dx$

(c) (2 marks) $\int_{-2}^{2}\sqrt{4-x^2}\,dx$, by using a geometric interpretation.

(d) (4 marks) $\int_{-2}^{2}\sqrt{4-x^2}\,dx$, by using the substitution $x = 2\sin t$, $-\pi/2 \le t \le \pi/2$

Q2. The growth rate $r(t)$ of a population $p(t)$ is given by

$$r(t) = \begin{cases} 100 - 6t, & 10 < t \le 20 \\ \dfrac{2t^2}{5}, & 0 \le t \le 10 \end{cases}$$

Assume, in addition, that $p(0) = 25$.

(a) (3 marks) Explain why $r(t)$ is integrable on $[0, 20]$.

(b) (3 marks) Determine when the population reaches its maximum.

(c) (4 marks) Find the maximum population in (b).

Q3. Consider the region bounded by $y = \dfrac{1}{x}$, the x-axis, and $x \geq 1$.

(a) (3 marks) Show that the area of the given region is infinity.

(b) (5 marks) If the region is rotated about the x-axis, a solid of revolution is obtained. Show that the volume of this solid is finite.

(c) (2 marks) Give an explanation why you have a finite number in one case while you have infinity in another case.

Q4. Follow the steps below to determine whether $\int_1^\infty e^{-x^2}\,dx$ converges or diverges:

Step 1 (4 marks) Explain why $f(u) = e^{-u}$ is decreasing on $[1, \infty)$. Use this fact to decide which function is larger on $[1, \infty)$—e^{-x} or e^{-x^2}?

Step 2 (4 marks) Determine whether $\int_1^\infty e^{-x}\,dx$ converges or diverges

Step 3 (2 marks) Based on the Comparison Test, make a conclusion about the convergence of $\int_1^\infty e^{-x^2}\,dx$

Q5. Solve $\dfrac{dy}{dt} = (y^3 + 4y)e^{-t}$.

Q6. The following model describes the interaction of an autotroph and its nutrient pool:

$$\begin{cases} \dfrac{dN}{dt} = N_I - aN - bNX + mX \\ \dfrac{dX}{dt} = bNX - (m+c)X \end{cases}$$

where N denotes total mass of nutrients; X biomass of autotroph; $N_I = 10$ the input rate of nutrients; $a = 3$, $c = 1$, and $m = 1$; $b > 0$ unknown. Note that an equilibrium of this system is a point (N, X) such that $\dfrac{dN}{dt} = 0$ and $\dfrac{dX}{dt} = 0$.

(a) (5 marks) Determine the scope of b so that the system has a nontrivial equilibrium.

(b) (5marks) Investigate how an increase in N (**Note:** X being fixed) at the point $(3, 3)$ would affect both $\dfrac{dN}{dt}$ and $\dfrac{dX}{dt}$

Hint: One way to do this is using partial derivatives

Q7. Assume $A = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}$

(a) (2 marks) Find the transpose, A^T, of A. Is it true that $A^T = A$?

(b) (4 marks) Compute A^TA and AA^T.

(c) (4 marks) Based on (b), answer: Is it true that $A^{-1} = A^T$? If NOT, use the row operation approach to find A^{-1}.

Q8. A linear system which contains a parameter a is given by

$$\begin{cases} x - 2y - 2z = 1 \\ -2x + y - 2z = 0 \\ -2x - 2y + a\,z = 0 \end{cases}$$

(a) (4 marks) Reduce the system into upper triangular form

(b) (2 marks) Determine for what value of a the system does not have a solution.

(c) (4 marks) Determine for what values of a the system has exactly one solution, and give the solution in the case of $a = -5$.

Q9. Assume $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

(a) (4 marks) Using paths $y = mx$, show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

(b) (2 marks) Note that $f(0, 0)$ is not defined. Question: Is it possible to assign a number to $f(0, 0)$ so that the function become continuous at that point?

(c) (4 marks) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Q10. Assume $F(x, y)$ is given by

$$F(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(a) (5 marks) Using the fact that $x^4 + y^4 \leq (x^2 + y^2)^2$, show that

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0$$

(b) (5 marks) Show that $F(x, y)$ is continuous at $(0, 0)$ and thus continuous everywhere.