BLUE PAPERS

SIMON FRASER UNIVERSITY

MATH 155
Final Examination

11 April 2008, 8:30–11:30

Last Name ____________________

Given Name(s) ____________________

Student # ____________________

Signature ____________________

INSTRUCTIONS

1. **Do not open this booklet until told to do so.**
2. Write your last name, given name(s), and student number in the box above. Sign on the last line of the box.
3. This exam has 11 questions on 11 pages. Check to make sure that your exam is complete.
4. No book, paper or device other than usual writing instruments, this examination booklet, and a scientific calculator are allowed. **Calculators with graphing and/or symbolic computation capabilities are not allowed.**
5. **During the examination, speaking to, communicating with, or exposing written papers to the view of other examinees is forbidden.**
6. You may use the **reverse side of the previous page** for rough work or if you run out of space.
7. **You may lose marks if your explanations are incomplete or poorly presented.**
8. **Stop writing when you are instructed to do so. Failure to follow instructions may result in penalties.**

Question	Maximum	Score
1	9	
2	9	
3	8	
4	9	
5	9	
6	8	
7	10	
8	10	
9	9	
10	9	
11	10	
Total	100	

[9] **1.** Find $\int_0^{\pi/4} \tan x\, dx$. Hint: Start by rewriting the integrand.

$$u = \cos x$$
$$du = -\sin x\, dx$$

$$\int_0^{\pi/4} \tan x \, dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} dx =$$

$$= \int_{\cos(0)}^{\cos(\pi/4)} \frac{-du}{u} = \Big[-\ln|u| \Big]_1^{\sqrt{2}/2} =$$

$$= -\ln \frac{\sqrt{2}}{2} - (-\ln 1) = -\ln 2^{-1/2} = \frac{\ln 2}{2}$$

$$\approx 0.34657$$

[9] **2.** Find $\int x^2 e^x \, dx$.

$\boxed{u = x^2, \ v' = e^x, \ v = e^x}$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx =$$

$\boxed{u = x, \ v' = e^x, \ v = e^x}$

$$= x^2 e^x - 2\left(x e^x - \int e^x \, dx\right) =$$

$$= x^2 e^x - 2x e^x + 2e^x + C = e^x(x^2 - 2x + 2) + C$$

[8] 3. If an object is propelled upward from ground level with the initial velocity 50 m/s, then its height at time t (seconds) is given by

$$s = -\frac{1}{2}gt^2 + 50t$$

where $g \approx 9.81$ m/s^2 is the gravitational acceleration. Find the average height of the object between times $t_0 = 2$ seconds and $t_1 = 4$ seconds.

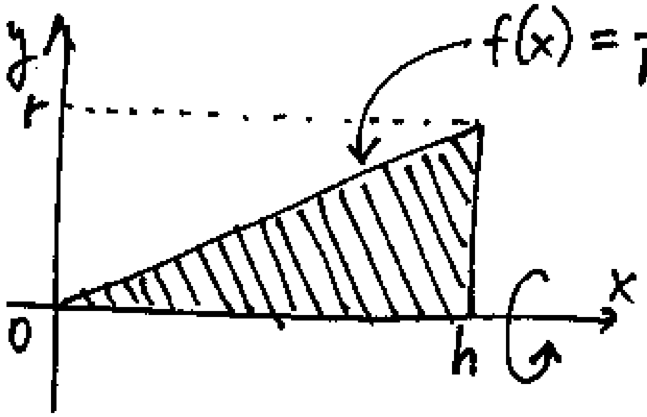
$$S_{avg} = \frac{1}{4-2}\int_2^4 \left(-\frac{1}{2}\cdot 9.81 t^2 + 50t\right) dt =$$

$$= \frac{1}{2}\left[\underbrace{-\frac{1}{6}\cdot 9.81}_{\approx -1.635}\cdot t^3 + 25t^2\right]_2^4 =$$

$$= \frac{1}{2}\left(-1.635\cdot 4^3 + 25\cdot 4^2 + 1.635\cdot 2^3 - 25\cdot 2^2\right) =$$

$$= 104.22$$

[9] **4.** Use the formula for the volume of a solid of revolution to find the volume of the right-circular cone with base radius r and height h.



$$V = \int_0^h \pi \left(\frac{r}{h}\cdot x\right)^2 dx = \pi \frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h =$$

$$= \pi \frac{r^2}{h^2}\left(\frac{h^3}{3} - 0\right) = \frac{\pi r^2 h}{3}$$

[9] 5. Evaluate $\int_1^{\infty} \frac{1}{\sqrt{x^3}}\,dx$.

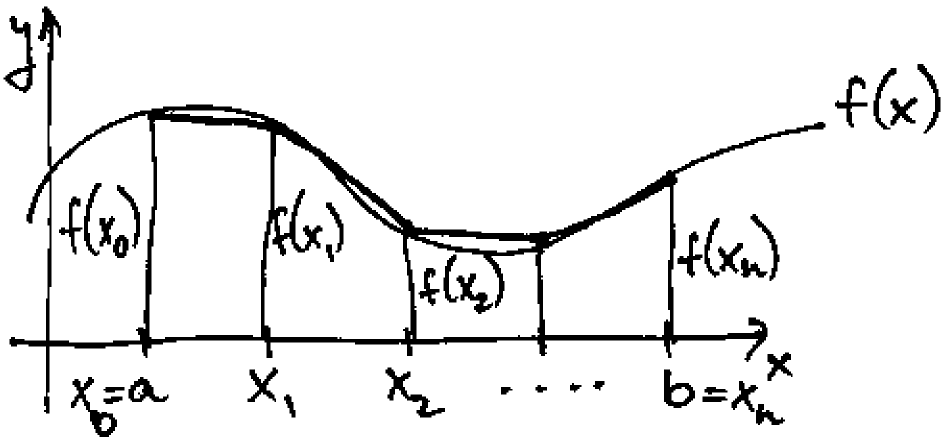
$$\int_1^{\infty} \frac{1}{\sqrt{x^3}}\,dx = \lim_{z\to\infty} \int_1^{z} \frac{dx}{\sqrt{x^3}} = \lim_{z\to\infty} \left[\frac{x^{-1/2}}{-\frac{1}{2}} \right]_1^{z} =$$

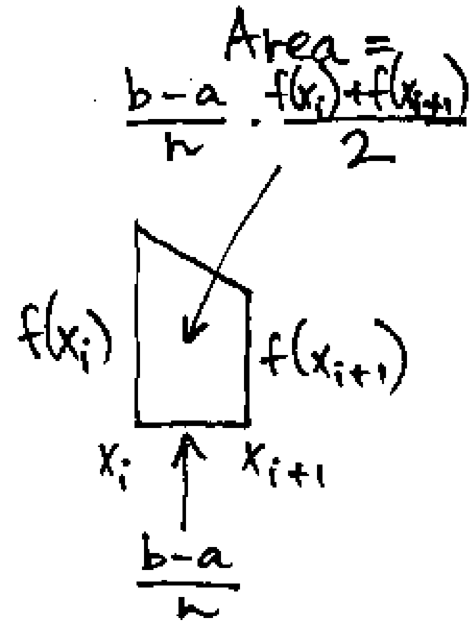
$$= \lim_{z\to\infty} \left(-2z^{-1/2} - \left(-2\cdot 1^{-1/2}\right)\right) =$$

$$= 0 + 2\cdot 1 = 2$$

[8] 6. Give the formula for the approximation of $\int_a^b f(x)\,dx$ using the Trapezoidal Rule. Explain clearly how this formula is obtained. Draw a picture.

The interval $[a,b]$ is subdivided into n intervals of equal length $\frac{b-a}{n}$. The integral $\int_a^b f(x)\,dx$ is approximated by the sum of areas of n trapezoids.





$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\cdots+\frac{f(x_{n-1})+f(x_n)}{2}\right)=$$

$$=\frac{b-a}{n}\left(\frac{f(x_0)}{2}+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{f(x_n)}{2}\right)$$

7. A tank of constant volume V contains a solution whose concentration at time t is denoted $C(t)$. The inflow rate into the pool is q (constant) and the outflow rate from the pool is also equal to q. The concentration of the inflowing solution is constant denoted C_I. The concentration of the outflowing solution at time t is the same as the concentration of the solution in the tank at time t, that is, $C(t)$. The concentration of the solution in the tank at time $t = 0$ is C_0.

[5] (a) Find a differential equation for the function $C(t)$. Explain how you obtained the equation.

- total mass of solute in the tank at time t is $C(t) \cdot V = CV$
- rate of change of solute mass = (rate at which mass enters) − (rate at which mass leaves)

$$\frac{d}{dt}(CV) = qC_I - qC$$

$$\frac{dC}{dt} = \frac{q}{V}(C_I - C)$$

[5] (b) Find $C(50)$ if $C_0 = 6\ \mathrm{g/m^3}$, $C_I = 2\ \mathrm{g/m^3}$, $q = 1\ \mathrm{m^3/s}$ and $V = 100\ \mathrm{m^3}$.

$$\frac{dC}{dt} = \frac{1}{100}(2 - C)$$

$$\frac{dC}{2-C} = \frac{dt}{100}$$

$$\int \frac{dC}{2-C} = \int \frac{dt}{100}$$

$$-\ln|2-C| = \frac{t}{100} + K_0$$

$$|2-C| = e^{-t/100 - K_0}$$

$$2 - C = K_1 \cdot e^{-\frac{t}{100}}$$

$$C(t) = 2 - K_1 \cdot e^{-\frac{t}{100}}$$

$$6 = 2 - K_1 \cdot e^{-0} \qquad (t=0)$$

$$C(t) = 2 + 4 \cdot e^{-\frac{t}{100}}$$

$$C(50) = 2 + 4 \cdot e^{-\frac{50}{100}} =$$

$$= 2 + 4 \cdot e^{-\frac{1}{2}} \approx 4.426$$

[5] 8. (a) Write down the differential equation for the *Logistic Growth* model. Explain the meaning of all symbols present in the model.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$N(t)$ ... population size at time t

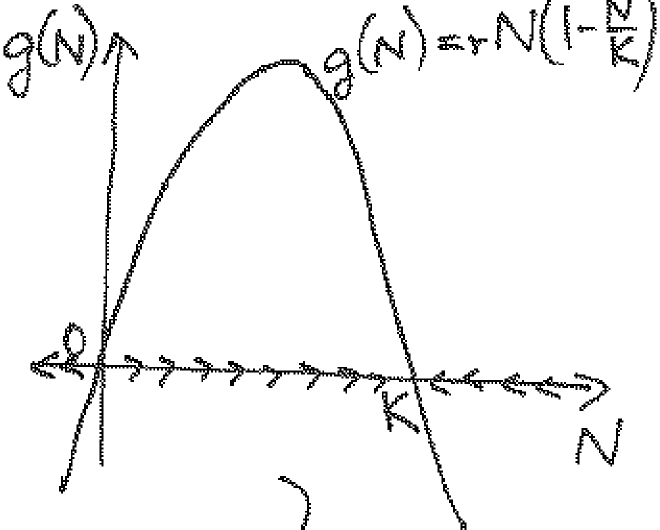
r ... intrinsic rate of growth (const. > 0)

K ... carrying capacity (const. > 0)

[5] (b) Find the equilibria for the *Logistic Growth* model, and analyze their stability.

$$rN\left(1 - \frac{N}{K}\right) = 0$$

$\hat{N}_1 = 0$ $\qquad$ $\hat{N}_2 = K$



$$g(N) = rN\left(1 - \frac{N}{K}\right)$$

$$g'(N) = r - \frac{2r}{K}N$$

$$g'(0) = r > 0$$

$$g'(K) = r - \frac{2r}{K}K = -r < 0$$

$\hat{N}_1 = 0$ is an unstable equilibrium

$\hat{N}_1 = K$ is a locally stable equilibrium

9. Denote by $p = p(t)$ the fraction of occupied patches in a metapopulation model, and assume that

$$\frac{dp}{dt} = cp(1-p) - p^2$$

where $c > 0$ is a constant. The value $\hat{p} = 0$ is a *trivial* equilibrium for this model.

[6] (a) Does there always exist a non-trivial equilibrium? If yes, find it and analyze its stability. Explain why we only consider equilibria in the interval $[0, 1]$. Does this condition hold for your solution?

$$cp(1-p) - p^2 = 0$$

$$p(c - p(c+1)) = 0 \Rightarrow \hat{p}_1 = 0, \quad \hat{p}_2 = \frac{c}{c+1}$$

Since $c>0$, we have $0 < \frac{c}{c+1} < 1$.

Since $p(t) \in [0,1]$ (it's the <u>fraction</u> of occupied patches), we're only interested in equilibria in $[0,1]$.

$$g(p) = cp(1-p) - p^2$$

$$g'(p) = c(1-p) - cp - 2p$$

$$g'\left(\frac{c}{c+1}\right) = c\left(1 - \frac{c}{c+1} - \frac{c}{c+1} - \frac{2}{c+1}\right) = c\left(-\frac{c+1}{c+1}\right) = -c < 0$$

$\frac{c}{c+1}$ is a locally stable equilibrium

[3] (b) Compare your result in part (a) with the corresponding results in Levins model. You do not need to analyze Levins model; you can just write down the results for it.

The result in part (a) differs from the corresponding result in Levins model.

In part (a) we found a nontrivial equilibrium for any $c > 0$. However in Levins model a nontrivial equilibrium exists only if $c > m$.

[9] 10. Suppose that the dynamics of a certain population is given by the following Leslie matrix:

$$L = \begin{pmatrix} 0 & 1.8 & 2.1 \\ 0.9 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$$

This matrix has an eigenvalue $\lambda = 1.5$. Set up a system of linear equations whose solution is a stable age distribution. Find the exact pattern of any stable age distribution.

Let $\begin{pmatrix} N_0 \\ N_1 \\ N_2 \end{pmatrix}$ be a stable age distribution.

$$\begin{pmatrix} 0 & 1.8 & 2.1 \\ 0.9 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix} \begin{pmatrix} N_0 \\ N_1 \\ N_2 \end{pmatrix} = 1.5 \begin{pmatrix} N_0 \\ N_1 \\ N_2 \end{pmatrix}$$

$$\begin{aligned} 1.8N_1 + 2.1N_2 &= 1.5N_0 \\ 0.9N_0 &= 1.5N_1 \\ 0.5N_1 &= 1.5N_2 \end{aligned}$$

$$\begin{aligned} -1.5N_0 + 1.8N_1 + 2.1N_2 &= 0 \\ 0.9N_0 - 1.5N_1 &= 0 \\ 0.5N_1 - 1.5N_2 &= 0 \end{aligned}$$

$$\begin{aligned} -1.5N_0 + 1.8N_1 + 2.1N_2 &= 0 \\ -0.42N_1 + 1.26N_2 &= 0 \\ 0.5N_1 - 1.5N_2 &= 0 \end{aligned}$$

$$\begin{aligned} -1.5N_0 + 1.8N_1 + 2.1N_2 &= 0 \\ -0.42N_1 + 1.26N_2 &= 0 \\ 0 &= 0 \end{aligned}$$

$N_2 = z \quad N_1 = \frac{1.26}{0.42} z = 3z$

$-1.5N_0 + 1.8(3z) + 2.1z = 0 \Rightarrow N_0 = \frac{7.5z}{1.5} = 5z$

Any vector $\begin{pmatrix} N_0 \\ N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} 5z \\ 3z \\ z \end{pmatrix}$ is a stable age distribution.

[10] 11. A rectangular box with edge lengths x, y, z has volume $V = xyz$ and surface area $S = 2(xy + xz + yz)$. Among all rectangular boxes of volume $V = 8\,\mathrm{m}^3$, find the one that has the smallest surface area. Use partial derivatives. Find the critical point(s) and discuss the nature of the critical point(s).

$V = xyz = 8 \Rightarrow z = \frac{8}{xy}$ $\quad x, y, z > 0$

$S = 2(xy + xz + yz) = 2\left(xy + \frac{8}{y} + \frac{8}{x}\right)$

$S_x = 2\left(y - \frac{8}{x^2}\right) \quad S_y = 2\left(x - \frac{8}{y^2}\right)$

$\left.\begin{array}{l} 2\left(y - \frac{8}{x^2}\right) = 0 \\ 2\left(x - \frac{8}{y^2}\right) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} y = \frac{8}{x^2} \\ x - \frac{8}{(8/x^2)^2} = 0 \Rightarrow x = \frac{x^4}{8} \end{array}$

$(x>0) \quad x^3 = 8$

$x = 2$

$y = \frac{8}{2^2} = 2$

Critical point $(x_0, y_0) = (2,2)$

$S_{xx} = \frac{32}{x^3} \quad S_{yy} = \frac{32}{y^3} \quad S_{xy} = 2$

$D = S_{xx}(2,2)\, S_{yy}(2,2) - (S_{xy}(2,2))^2 =$

$= \frac{32}{2^3} \cdot \frac{32}{2^3} - 2^2 = 4 \cdot 4 - 4 = 12 > 0$

$S_{xx}(2,2) = \frac{32}{2^3} = 4 > 0$ $\longrightarrow$ S has a local minimum at $(2,2)$.

Since there are no other critical points, the box with the smallest surface area has edge lengths $x = 2,\ y = 2,\ z = 8/(2\cdot 2) = 2$ (cube).