ORANGE PAPERS

SIMON FRASER UNIVERSITY

MATH 155 Final Examination

12 April 2007, 08:30–11:30

Last Name _____

Given Name(s) _____

Student # _____

Signature _____

INSTRUCTIONS

1. **Do not open this booklet until told to do so.**
2. Write your last name, given name(s), and student number in the box above. Sign on the last line of the box.
3. This exam has 12 questions on 13 pages. Check to make sure that your exam is complete.
4. No book, paper or device other than usual writing instruments, this examination booklet, and a scientific calculator are allowed. **Calculators with graphing and/or symbolic computation capabilities are not allowed.**
5. **During the examination, speaking to, communicating with, or exposing written papers to the view of other examinees is forbidden.**
6. You may use the **reverse side of the previous page** for rough work or if you run out of space.
7. **You may lose marks if your explanations are incomplete or poorly presented.**
8. **Stop writing when you are instructed to do so. Failure to follow instructions may result in penalties.**

Question	Maximum	Score
1	8	
2	8	
3	6	
4	9	
5	8	
6	6	
7	8	
8	11	
9	8	
10	9	
11	11	
12	8	
Total	100	

[2] 1. (a) Use the substitution $u = \sin x$ to find $\int \sin x \cos x \, dx$.

$du = \cos x \, dx$

$$\int \sin x \cos x \, dx = \int u \, du = \frac{u^2}{2} + C_1 = \boxed{\frac{(\sin x)^2}{2} + C_1}$$

[2] (b) Use the substitution $u = \cos x$ to find $\int \sin x \cos x \, dx$.

$du = -\sin x$

$$\int \sin x \cos x \, dx = -\int u \, du = -\frac{u^2}{2} + C_2 = \boxed{-\frac{(\cos x)^2}{2} + C_2}$$

[4] (c) The antiderivatives obtained in parts (a) and (b) are different. Write down the general relation satisfied by any two antiderivatives of the same function. Verify that this relation holds for the antiderivatives obtained in parts (a) and (b) above.

If $F'(x) = G'(x)$ then $F(x) - G(x) = \text{Const.}$

For (a) and (b):

$$\frac{(\sin x)^2}{2} + C_1 - \left[-\frac{(\cos x)^2}{2} + C_2\right] =$$

$$= \frac{1}{2}\underbrace{\left[(\sin x)^2 + (\cos x)^2\right]}_{1} + C_1 - C_2 = \frac{1}{2} + C_1 - C_2 = \text{const.} \checkmark$$

[8] 2. Find $\int e^x \sin x \, dx$.

$u = e^x \quad v' = \sin x \quad v = -\cos x$

$$\int e^x \sin x \, dx = e^x(-\cos x) - \int e^x(-\cos x)\, dx =$$

$$= -e^x \cos x + \int e^x \cos x \, dx =$$

$u = e^x$
$v' = \cos x$
$v = \sin x$

$$= -e^x \cos x + \left[e^x \sin x - \int e^x \sin x \, dx \right] =$$

$$= e^x(-\cos x + \sin x) - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x(-\cos x + \sin x) + C$$

[6] **3.** Suppose that h hours after midnight, the outside air temperature (in degrees Celsius) can be modeled by the formula

$$T(h) = 6 - \frac{1}{5}(h-13)^2.$$

Find the average temperature between 5:00 A.M. and 2:00 P.M.

at 5am ... $h=5$

at 2pm ... $h=14$

$$T_{avg} = \frac{1}{14-5}\int_5^{14}\left[6-\frac{1}{5}(h-13)^2\right]dh =$$

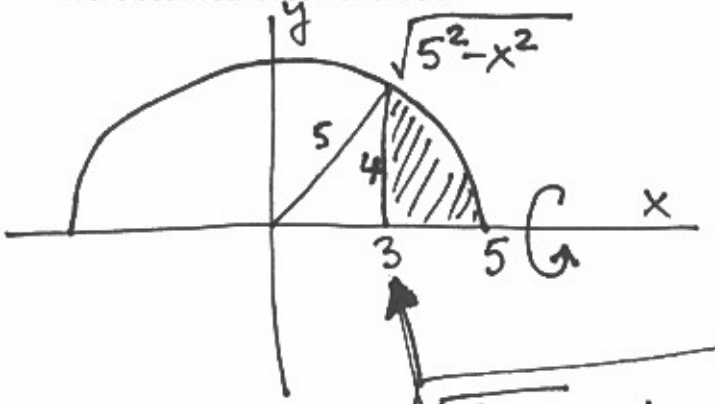
$$= \frac{1}{9}\int_{-8}^{1}\left[6-\frac{u^2}{5}\right]du =$$

$$= \frac{1}{9}\left[6u-\frac{u^3}{15}\right]_{-8}^{1} = \frac{1}{9}\left(6-\frac{1}{15}+48-\frac{8^3}{15}\right) =$$

$$= \frac{1}{9}\left(54-\frac{513}{15}\right) = 6-\frac{57}{15} = 6-\frac{19}{5} =$$

$$\boxed{= \frac{11}{5} = 2.2\ ^\circ C}$$

[9] **4.** An orange has the shape of a perfect sphere (ball) of radius 5 cm. Suppose that we divide the orange into two pieces by a planar cut; the radius of the circle created by the cut is 4 cm. Use an integral to determine the volume of the smaller of the two pieces (a so-called *spherical cap*) that we obtained after the cut.



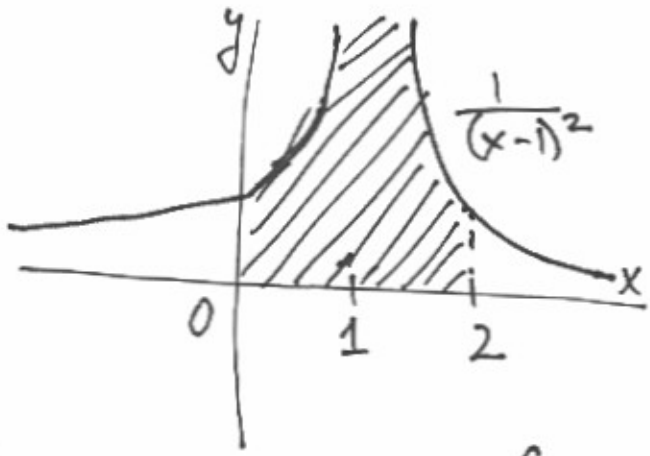
$\sqrt{5^2-x^2} = 4 \Rightarrow x = 3$

$$\text{Volume} = \int_3^5 \pi \left[\sqrt{5^2-x^2}\right]^2 dx = \int_3^5 \pi (25-x^2)\,dx =$$

$$= \pi \left[25x - \frac{x^3}{3}\right]_3^5 = \pi\left(25\cdot 5 - \frac{125}{3} - 25\cdot 3 + \frac{27}{3}\right) =$$

$$= \pi\left(50 - \frac{98}{3}\right) = \boxed{\frac{52}{3}\pi \text{ cm}^3 \approx 54.5 \text{ cm}^3}$$

[8] 5. Evaluate $\int_0^2 \frac{1}{(x-1)^2}\,dx$.



$$\int_0^2 \frac{dx}{(x-1)^2} = \lim_{c\to 1^-} \int_0^c \frac{dx}{(x-1)^2} + \lim_{c\to 1^+} \int_c^2 \frac{dx}{(x-1)^2} =$$

$$= \lim_{c\to 1^-} \int_{-1}^{c-1} \frac{du}{u^2} + \lim_{c\to 1^+} \int_{c-1}^{1} \frac{du}{u^2} =$$

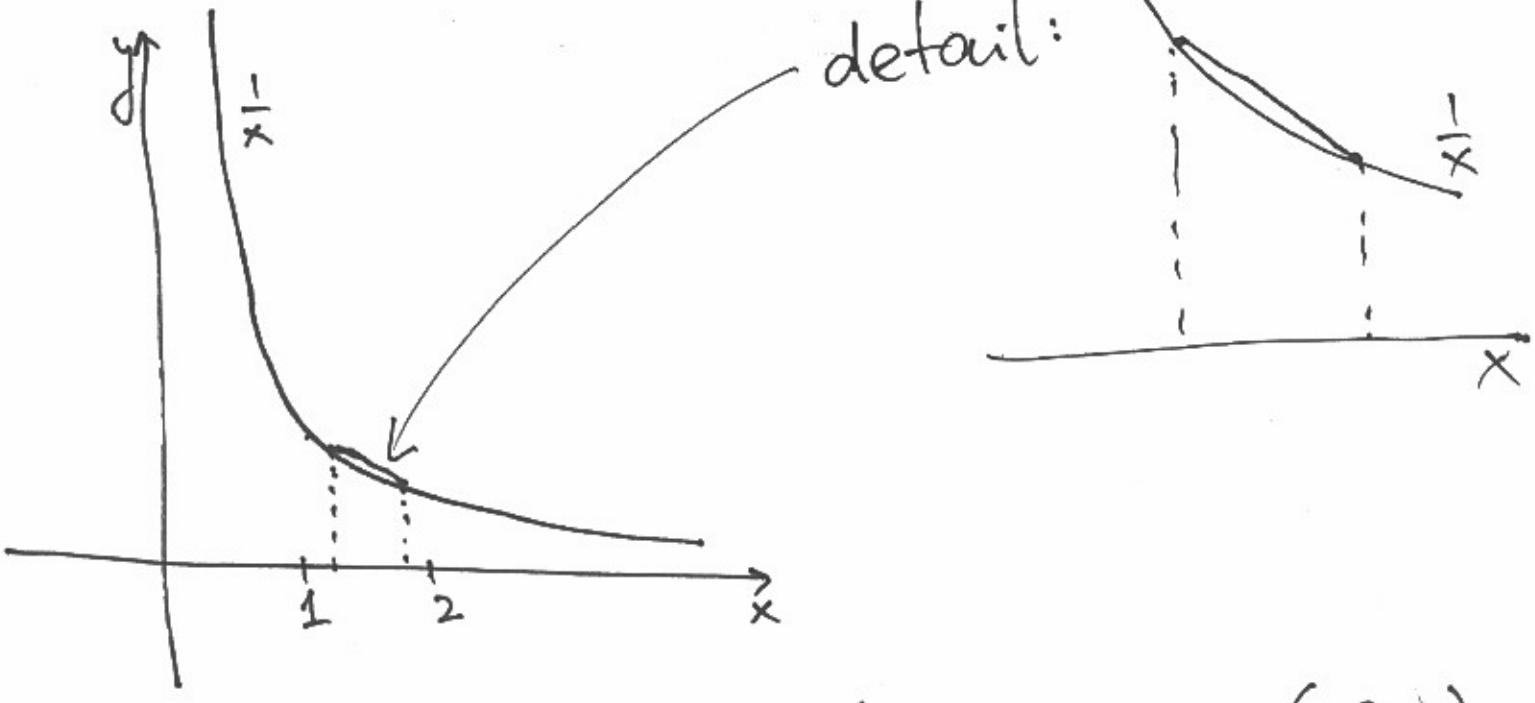
$$= \lim_{c\to 1^-} \left[-\frac{1}{u}\right]_{-1}^{c-1} + \lim_{c\to 1^+} \left[-\frac{1}{u}\right]_{c-1}^{1} =$$

$$= \underbrace{\lim_{c\to 1^-} \left(-\frac{c}{c-1}\right)}_{\infty} + \underbrace{\lim_{c\to 1^+} \left(-\frac{c-2}{c-1}\right)}_{\infty}$$

Therefore $\int_0^2 \frac{dx}{(x-1)^2}$ diverges.

[6] 6. Suppose that you found an approximation of $\int_1^2 \frac{dx}{x}$ using the Trapezoidal Rule with $n = 1,000$ intervals. Is this approximate value of the integral *larger* or *smaller* than the exact value of the integral? Explain your answer. You are *not* allowed to perform any actual numerical computations with the Trapezoidal Rule.

Hint: Draw a graph!



Because the function $y = \frac{1}{x}$ is concave up (p.265), as can be seen from the graph or by considering $y'' = \frac{2}{x^3} > 0$ for $x > 0$, it follows that the upper edges of all trapezoids lie above the graph $y = \frac{1}{x}$ and they touch the graph at the endpoints. Therefore the approximation found by the Trapezoidal Rule is larger than the exact value of $\int_1^2 \frac{dx}{x}$.

[8] **7.** Compute an approximation of $\ln(0.9)$ with the accuracy of three places after the decimal point. You are only allowed to use the basic arithmetic operations $+, -, \times, \div$.

Hint: Use Taylor polynomials of increasing degree. Stop as soon as two successive approximations (computed to the full precision of your calculator) agree after being rounded to the target precision. Show also the values before rounding.

We'll use Taylor polynomials about $x=1$ for $f(x)=\ln x$.

$f(1)=0$

$f'(x)=\frac{1}{x}$ $f'(1)=1$ $P_1(x)=0+1\cdot(x-1)=x-1$

$P_1(0.9)=-0.1$

$f''(x)=-\frac{1}{x^2}$ $f''(1)=-1$ $P_2(x)=P_1(x)+\frac{-1}{2!}(x-1)^2=x-1-\frac{1}{2}(x-1)^2$

$P_2(0.9)=-0.105$ ← same rounded value

$f'''(x)=\frac{2}{x^3}$ $f'''(1)=2$ $P_3(x)=P_2(x)+\frac{2}{3!}(x-1)^3$

$P_3(0.9)=-0.105+\frac{1}{3}\cdot(-0.1)^3=$

$=-0.1053333\ldots\approx-0.105$

$$\ln(0.9)\approx-0.105$$

8. The rate at which the temperature of an object changes is proportional to (i.e., it is a constant multiple of) the difference between its own temperature and the temperature of the surrounding medium.

[5] (a) Let $T(t)$ denote the temperature of the object at time t. Let the constant of proportionality be denoted k. Assume that the temperature of the surrounding medium is constant and denoted by M. Describe the situation given in the previous paragraph by a differential equation. Indicate whether in your model the constant k is positive or negative.

$$\frac{dT}{dt} = k(M-T), \quad k>0$$

or $$\frac{dT}{dt} = k(T-M), \quad k<0$$

[6] (b) Assume that $T(0) = 80$, $M = 30$ and $k = 0.2$ or $k = -0.2$ depending on the choice that you made in part (a). Find $T(10)$, i.e. the temperature of the object at time $t = 10$.

$$\frac{dT}{dt} = 0.2(30-T)$$

$$\frac{dT}{6-0.2T} = dt$$

$$\int \frac{dT}{6-0.2T} = \int dt$$

$$-5\ln|6-0.2T| = t + C_1$$

$$\ln|6-0.2T| = -\frac{t}{5} + C_1$$

$$6-0.2T = C\cdot e^{-\frac{t}{5}}$$

$$T(t) = 30 - C\cdot e^{-\frac{t}{5}}$$

at $t=0$

$$80 = 30 - C$$

$$C = -50$$

$$T(t) = 30 + 50e^{-\frac{t}{5}}$$

$$T(10) = 30 + 50\cdot e^{-2}$$

$$\approx 36.8$$