MATH 155
FINAL

8:30-11:30, April 14, 2004
Instructor: Peter Berg

Family name: ____________ **Initials:** ______
Student ID number: ____________
Signature: ____________

READ INSTRUCTIONS CAREFULLY:

- **Do not lift the cover page until instructed!**
- Fill out your name and ID in the space provided and sign.
- You MUST NOT use a calculator. NO other aids.
- Answer all questions, explaining your answers carefully in the space provided. If you run out of space, use the back of the **preceding** page.
- This exam consists of 10 questions and 11 pages (including this one).

Question	1	2	3	4	5	
Grade	/10	/10	/10	/10	/10	
	6	7	8	9	10	Total
	/10	/10	/10	/10	/10	/100

Good Luck!

1. (10 marks) *Battle of Orders: Logarithm vs Polynomial*

   Evaluate
   $$\int_1^\infty \frac{\ln x}{x^2}\,dx$$
   and sketch the graph of the integrand over the range of integration.

   Solution: Substitute $v = \ln x$, $v' = 1/x$, $w' = 1/x^2$, $w = -1/x$. Then integration by parts gives

   $$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x}\bigg|_1^\infty + \int_1^\infty \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\bigg|_1^\infty \\ &= 1.\end{aligned}$$

   Here, we used L'Hospital's rule to find

   $$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

   Graph.

2. (10 marks) *Slicing through Bert's head*

The straight line $y = 1 - x$ breaks the interior of the ellipse

$$2x^2 + y^2 = 1$$

into two parts. What is the size of the smaller area?

You might want to know that

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \arcsin\left(\frac{x}{a}\right)\right]$$

Solution: First, we determine the points of intersection of the curves. Substitution of $y = 1 - x$ into the equation for ellipse yields

$$2x^2 + (1 - x)^2 = 1$$

or

$$x(3x - 2) = 0.$$

Hence, they intersect at $x = 0$ and $x = \frac{2}{3}$. The area of interest is

$$\begin{aligned}
& \int_0^{2/3} [\sqrt{1 - 2x^2} - (1 - x)]\, dx \\
= & \int_0^{2/3} [\sqrt{2}\sqrt{\frac{1}{2} - x^2} - 1 + x]\, dx \\
= & \frac{\sqrt{2}}{2}\left[x\sqrt{\frac{1}{2} - x^2} + \frac{1}{2}\arcsin\left(\sqrt{2}x\right)\right] - x + \frac{x^2}{2}\Bigg|_0^{2/3} \\
= & \frac{1}{2\sqrt{2}} \arcsin\left(\frac{2\sqrt{2}}{3}\right) - \frac{1}{3} \\
(= & \; 0.1019)
\end{aligned}$$

where we set $a^2 = \frac{1}{2}$.

3. (10 marks) *AWOL in BC*

The population dynamics of a little gulf island is described by

$$\frac{dN}{dt} = tN + 4t$$

where $N(0) = 100$. What is $N(t)$ for $t \geq 0$? What is the population size after $t = 1$ (year)?

Solution: Separation of variables and use of initial condition yields

$$\int_{100}^{N} \frac{1}{\bar{N} + 4}\, d\bar{N} = \int_{0}^{t} \bar{t}\, d\bar{t}.$$

We know that $N > 0$ for all t and $t \geq 0$. Hence, straightforward integration gives

$$\begin{aligned}
\int_{100}^{N} \frac{1}{\bar{N} + 4}\, d\bar{N} &= \int_{0}^{t} \bar{t}\, d\bar{t} \\
\Rightarrow \ln(\bar{N} + 4)\Big|_{100}^{N} &= \frac{1}{2}\bar{t}^2\Big|_{0}^{t} \\
\Rightarrow \ln\left(\frac{N + 4}{104}\right) &= \frac{1}{2}t^2 \\
\Rightarrow N(t) &= 104 \exp\left(\frac{t^2}{2}\right) - 4
\end{aligned}$$

We find a population of

$$N(1) = 104\sqrt{e} - 4 \quad (\approx 167)$$

after one year.

4. (10 marks) *3 times 3 times 3 times 3*

Let us consider the following matrices

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 2 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, C = \begin{pmatrix} 7 \\ -5 \\ 7 \end{pmatrix}.$$

Compute BA and solve $AX = C$.

Solution:

$$BA = \begin{pmatrix} 3 & 3 & -7 \\ 4 & 3 & 4 \\ 2 & -5 & 2 \end{pmatrix}$$

We use Gaussian elimination to determine X

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & -3 & 1 & -5 \\ 2 & 1 & 0 & 7 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & -3 & 1 & -5 \\ 0 & 1 & 4 & -7 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & -3 & 1 & -5 \\ 0 & 3 & 12 & -21 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 13 & -26 \end{array}\right).$$

Therefore, $z = -2 \Rightarrow y = 1 \Rightarrow x = 3$.

5. (10 marks) *Think positive and negative and positive.*

What is the area between $x = 0$, $x = 3$, $y = 0$ and the function

$$f(x) = x^2 - 3x + 2 \quad ?$$

Solution: We need to know the roots of $f(x)$. Setting

$$f(x) = 0,$$

we find the roots $x = 1$, $x = 2$. Therefore, the area is

$$\begin{aligned}
& \int_0^1 (x^2 - 3x + 2)\,dx - \int_1^2 (x^2 - 3x + 2)\,dx + \int_2^3 (x^2 - 3x + 2)\,dx \\
= \;& \left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right)\Bigg|_0^1 - \left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right)\Bigg|_1^2 + \left(\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right)\Bigg|_2^3 \\
= \;& 2\left(\frac{1}{3} - \frac{3}{2} + 2\right) + 2\left(\frac{8}{3} - 6 + 4\right) + \left(9 - \frac{27}{2} + 6\right) \\
= \;& \frac{5}{3} - \frac{4}{3} + \frac{3}{2} \\
= \;& \frac{11}{6}.
\end{aligned}$$

6. (10 marks) *C'mon!*

i) What is

$$\frac{d^2}{dx^2} \int \sin(x^2)\, dx \quad ?$$

Solution:

$$\frac{d^2}{dx^2} \int \sin(x^2)\, dx = 2x \cos(x^2)$$

ii) What is

$$\int \frac{d^2}{dx^2} (\sin x)^2\, dx \quad ?$$

Solution:

$$\int \frac{d^2}{dx^2} (\sin x)^2\, dx = 2 \sin(x) \cos(x) + C$$

where C is a real constant.

7. (10 marks) *The infinite tear drop*

We look from the side onto an infinite tear drop with a cross-sectional area described by

$$f_{\pm}(x) = \pm \frac{\sqrt{x}}{(x+1)^{3/2}} \ .$$

What is the volume of the tear drop, whose cross-sectional area is circular when we slice through it perpendicular to the x–axis? What is the radius of the tear drop when it assumes a spherical shape?

Solution: We compute the volume as

$$\begin{aligned} & \pi \int_0^\infty f_+^2 \, dx \\ = \ & \pi \int_0^\infty \frac{x}{(x+1)^3} \, dx \end{aligned}$$

We use partial fractions and write

$$\frac{x}{(x+1)^3} = \frac{A}{(x+1)^2} + \frac{B}{(x+1)^3}$$

and find $A = 1$, $B = -1$. Hence, the integral becomes

$$\begin{aligned} \pi \int_0^\infty \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3} \right) dx \ &= \ \pi \left(-\frac{1}{x+1} + \frac{1}{2(x+1)^2} \right) \Bigg|_0^\infty \\ &= \ \frac{\pi}{2}. \end{aligned}$$

Assuming a shperical shape, the radius R of the tear drop would be

$$\frac{4}{3}\pi R^3 = \frac{\pi}{2} \Rightarrow R = \left(\frac{3}{8} \right)^{1/3} .$$

8. (10 marks) *Positive, negative or zero?*

Sketch the graph of the function

$$f(x) = \exp(-x)\ \sin(x)$$

and compute its integral for $x \in [0;\infty)$.

Solution: We use integration by parts twice (two ways) and compute

$$\begin{aligned}
&\int_0^\infty \exp(-x)\sin(x)\,dx \\
= \;& -\exp(-x)\sin(x)|_0^\infty + \int_0^\infty \exp(-x)\cos(x)\,dx \\
= \;& -\exp(-x)(\sin(x)+\cos(x))|_0^\infty - \int_0^\infty \exp(-x)\sin(x)\,dx
\end{aligned}$$

Bringing the last term on the right-hand side over to the left-hand side and division by two, we obtain

$$\int_0^\infty \exp(-x)\sin(x)\,dx = -\frac{1}{2}\exp(-x)(\sin(x)+\cos(x))\Big|_0^\infty = \frac{1}{2}.$$

Graph.

9. (10 marks) *Two dead seas, one sea level and two endless walls to climb.*

Find the equilibria of the differential equation

$$\frac{dy}{dx} = y^4 - y^3 - 2y^2.$$

What is the linear stability of these equilibria? (Use eigenvalues!) Graph $\frac{dy}{dt}$ as a function of y and discuss local stability of the centre equilibrium and global stability of all equilibria.

Solution: Setting the right-hand side to zero, we find three roots and, hence, equilibria

$$\hat{y} = 0, \hat{y} = -1, \hat{y} = 2.$$

Defining $g(y) = y^4 - y^3 - 2y^2$ ($\Rightarrow g'(y) = 4y^3 - 3y^2 - 4y$), we find the corresponding eigenvalues to be

$$\begin{aligned} g'(0) = 0 &\quad , \quad \text{stability unclear} \\ g'(-1) = -3 &\quad , \quad \text{linear stable} \\ g'(2) = 12 &\quad , \quad \text{linear unstable.} \end{aligned}$$

Graph.

From the graph we see that $\hat{y} = 0$ is an unstable equilibrium. Moreover, $\hat{y} = -1$ is not globally stable since a perturbation past the point $\hat{y} = 2$ would lead to runaway values of y. Therefore, none of the equilibria is globally stable.

10. (10 marks) *Turning logarithms into polynomials*

Determine the Taylor polynomial of order 5, $P_5(x)$, of the function $y(x) = \ln(x)$ about $x = 1$. What is $y(e)$? Is $P_5(e)$ less or greater than $y(e)$? (Use inequalities!)

Solution: First we determine the derivatives at $x = 1$ $(f(x) = y(x) = \ln x)$:

$$\begin{aligned}
f(x) &= \ln x &\Rightarrow& \quad f(1) = 0 \\
f'(x) &= \frac{1}{x} &\Rightarrow& \quad f'(1) = 1 \\
f''(x) &= -\frac{1}{x^2} &\Rightarrow& \quad f''(1) = -1 \\
f^{(iii)}(x) &= \frac{2}{x^3} &\Rightarrow& \quad f^{(iii)}(1) = 2 \\
f^{(iv)}(x) &= \frac{-6}{x^4} &\Rightarrow& \quad f^{(iv)}(1) = -6 \\
f^{(v)}(x) &= \frac{24}{x^5} &\Rightarrow& \quad f^{(v)}(1) = 24
\end{aligned}$$

Hence, the Taylor polynomial is

$$\begin{aligned}
P_5(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(iii)}}{6}(x-1)^3 \\
&\quad + \frac{f^{(iv)}}{24}(x-1)^4 + \frac{f^{(v)}}{120}(x-1)^5 \\
&= -1 + x - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5.
\end{aligned}$$

We estimate

$$\begin{aligned}
P_5(e) &> 1.7 - \frac{1}{2}1.8^2 + \frac{1}{3}1.7^3 - \frac{1}{4}1.8^4 + \frac{1}{5}1.7^5 \\
&> 1.0 \quad \text{after some calculation with pencil and paper}
\end{aligned}$$

Here, we used $1.8 > e - 1 > 1.7$. Therefore, $P_5(e) > y(e) = 1$.