BLUE PAPERS

SIMON FRASER UNIVERSITY

MATH 155 Final Exam

23 April 2010, 15:30–18:30

Last Name ____________________

Given Name(s) ____________________

Student # ____________________

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Circle section number: D100 D400

Signature ____________________

INSTRUCTIONS

1. **Do not open this booklet until told to do so.**
2. Fill out the box in the upper right corner of this page.
3. This exam has 11 questions on 11 pages. Check to make sure that your exam is complete.
4. No book, paper or device other than usual writing instruments, this examination booklet, and a scientific calculator are allowed. **Calculators with graphing and/or symbolic computation capabilities are not allowed.**
5. **During the examination, speaking to, communicating with, or exposing written papers to the view of other examinees is forbidden.**
6. You may use the **reverse side of the previous page** for rough work or if you run out of space.
7. **You may lose marks if your explanations are incomplete or poorly presented.**
8. **Stop writing when you are instructed to do so. Failure to follow instructions may result in penalties.**

Question	Maximum	Score
1	8	
2	8	
3	10	
4	9	
5	11	
6	9	
7	8	
8	10	
9	9	
10	8	
11	10	
Total	100	

[8] **1.** Evaluate $\int_0^1 xe^{-x^2}\,dx$.

$$\int_0^1 xe^{-x^2}\,dx =$$

$$\boxed{\begin{aligned} u &= -x^2 \\ \frac{du}{dx} &= -2x \\ dx &= \frac{du}{-2x} \end{aligned}}$$

$$= \int_{-0^2}^{-1^2} x e^u \frac{du}{-2x} =$$

$$= -\int_{-1}^{0} -\frac{1}{2} e^u\,du = \frac{1}{2}\int_{-1}^{0} e^u\,du = \frac{1}{2}\left[e^u\right]_{-1}^{0} =$$

$$= \frac{1}{2}\left(1 - e^{-1}\right) \approx 0.316$$

[8] **2.** Evaluate $\int_{-2}^{0} xe^{-x}\,dx$.

$$\boxed{u = x, \quad v' = e^{-x}, \quad v = -e^{-x}}$$

$$\int_{-2}^{0} xe^{-x}\,dx = \left[x(-e^{-x})\right]_{-2}^{0} - \int_{-2}^{0} -e^{-x}\,dx =$$

$$= 0 - (-2)(-e^{2}) + \int_{-2}^{0} e^{-x}\,dx =$$

$$= -2e^{2} + \left[-e^{-x}\right]_{-2}^{0} = -2e^{2} - 1 - (-e^{2}) =$$

$$= -e^{2} - 1 \approx -8.389$$

[10] **3.** Evaluate $\displaystyle\int \frac{8x^2+4x-6}{4x^2+1}\,dx$.

$$\frac{8x^2+4x-6}{4x^2+1} = 2 + \frac{4x-8}{4x^2+1} \quad \text{(long division)}$$

$$\int \frac{8x^2+4x-6}{4x^2+1}\,dx = \int 2\,dx + \int \frac{4x-8}{4x^2+1}\,dx =$$

$$= 2x + \int \frac{2u-8}{u^2+1}\,\frac{du}{2} =$$

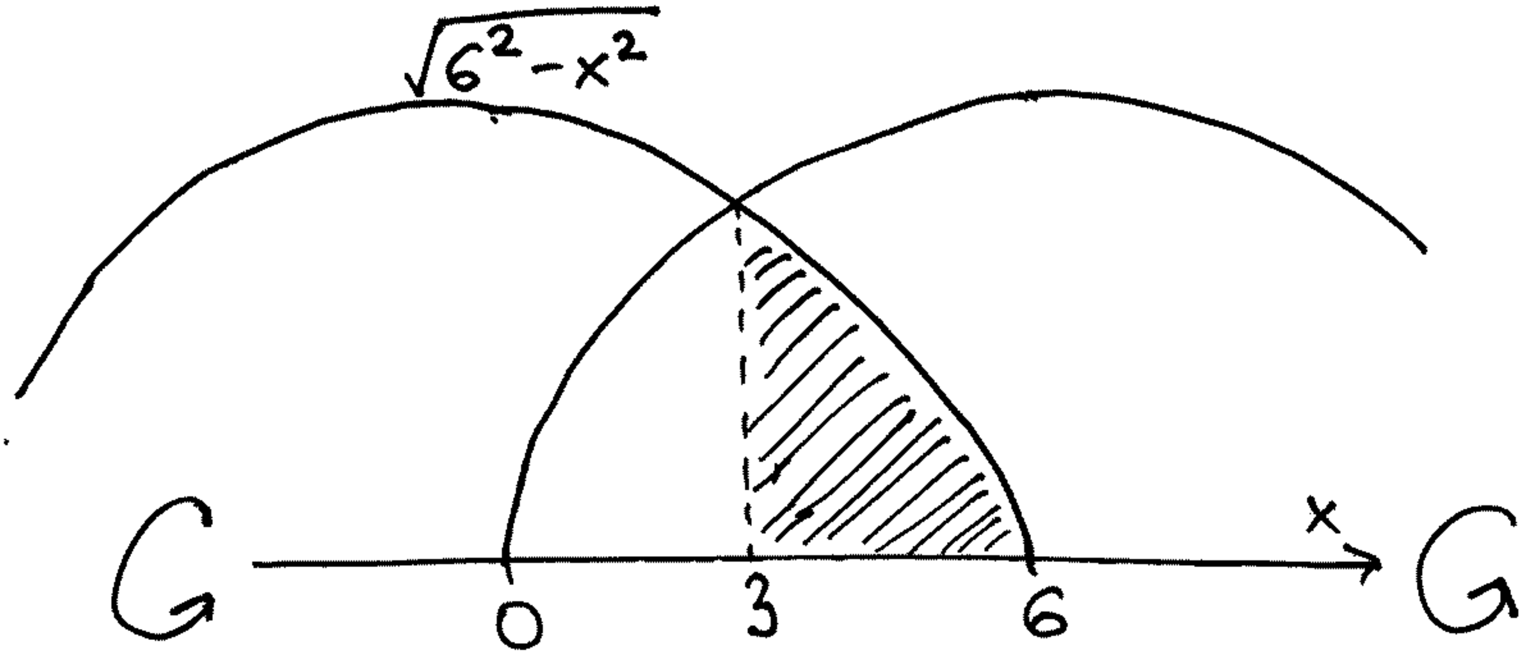
$$\boxed{\begin{aligned} u &= 2x \\ \frac{du}{dx} &= 2 \\ dx &= \frac{du}{2} \end{aligned}}$$

$$= 2x + \int \frac{1}{2}\cdot\frac{2u}{u^2+1}\,du - 4\int \frac{du}{u^2+1} =$$

$$= 2x + \frac{1}{2}\ln|u^2+1| - 4\tan^{-1}u + C =$$

$$= 2x + \frac{1}{2}\ln|4x^2+1| - 4\tan^{-1}(2x) + C$$

[9] 4. Compute the volume of the intersection of two spheres of radius 6 cm that are positioned such that the distance of their centres is 6 cm. (Hint: Place the centre of one of the spheres in the origin, and split the intersection of the spheres into two symmetric parts.)



The volume of one half of the intersection is

$$\int_3^6 \pi\left(\sqrt{6^2-x^2}\right)^2 dx = \pi\int_3^6 36-x^2\,dx =$$

$$= \pi\left[36x - \frac{x^3}{3}\right]_3^6 = \pi\left(216 - \frac{216}{3} - 108 + \frac{27}{3}\right) =$$

$$= 45\pi \text{ cm}^3 \approx 141.37 \text{ cm}^3.$$

The volume of the intersection is

$$2\cdot(45\pi \text{ cm}^3) = 90\pi \text{ cm}^3 \approx 282.74 \text{ cm}^3.$$

5. Suppose that at time t an object has temperature $T(t)$. The object is brought into a room that is kept at a constant temperature T_a. Newton's law of cooling states that T satisfies the differential equation

$$\frac{dT}{dt} = k(T - T_a)$$

where k is a negative constant.

[6] (a) Suppose that at time $t = 0$ we bring an object whose temperature is 28° C in a room whose temperature is 20° C. Solve for T (note that k will remain undetermined in this part of the question).

$$\frac{dT}{dt} = k(T-20)$$

$$\frac{dT}{T-20} = k\,dt$$

$$\int \frac{dT}{T-20} = \int k\,dt$$

$$\ln|T-20| = kt + C$$

$$|T-20| = e^{kt+C}$$

$$T-20 = \pm e^{C} \cdot e^{kt}$$

$$T = 20 + C_1 \cdot e^{kt}$$

at $t=0$:

$$28 = 20 + C_1 \Rightarrow C_1 = 8$$

$$T(t) = 20 + 8e^{kt}$$

[5] (b) Assume the situation introduced in part (a). Suppose that it takes the object 40 minutes to cool to 25° C. How long will it take the object to cool to 22° C?

$$25 = 20 + 8e^{k \cdot 40}$$

$$e^{40k} = \frac{5}{8}$$

$$40k = \ln\frac{5}{8}$$

$$k = \frac{1}{40}\ln\frac{5}{8}$$

$$\approx -0.0118$$

$$22 = 20 + 8e^{\frac{\ln 5/8}{40}t}$$

$$e^{\frac{\ln 5/8}{40}t} = \frac{1}{4}$$

$$\frac{\ln 5/8}{40}t = \ln\frac{1}{4}$$

$$t = \frac{\left(\ln\frac{1}{4}\right)\cdot 40}{\ln 5/8}$$

$$\approx 117.98 \text{ minutes}$$

[9] **6.** Solve the differential equation

$$\frac{dy}{dx} = e^{-3y}$$

where $y(0) = 0$.

$$\frac{dy}{dx} = e^{-3y}$$

$$\frac{dy}{e^{-3y}} = dx$$

$$\int e^{3y}\,dy = \int dx$$

$$\frac{1}{3}\cdot e^{3y} = x + C$$

$$e^{3y} = 3x + C_1$$

$(x = 0)$ $$e^{3\cdot 0} = 3\cdot 0 + C_1 \Rightarrow C_1 = 1$$

$$e^{3y} = 3x + 1$$

$$3y = \ln(3x+1)$$

$$y = \frac{1}{3}\ln(3x+1)$$

[8] **7.** Compute the Taylor polynomial of degree 3 about $x = 0$ for $f(x) = e^x$. Use your result to compute an approximate value of $e^{0.3}$. Compare your approximation with the exact value of $e^{0.3}$.

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$f'''(x) = e^x \qquad f'''(0) = 1$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 =$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_3(0.3) = 1 + 0.3 + \frac{0.3^2}{2} + \frac{0.3^3}{6} = 1.3495$$

$$e^{0.3} = 1.349858\ldots$$

[3] **8.** (a) Write the following system of linear equations in the matrix form:

$$\begin{array}{rrrcr} x_1 & & -x_3 & = & 2 \\ x_1 & +2x_2 & & = & -3 \\ & -x_2 & -x_3 & = & 4 \end{array}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}}_{B}$$

$$AX = B \Rightarrow X = A^{-1}B$$

[7] (b) Use the inverse matrix to solve the system of linear equations given in part (a).

Compute A^{-1}. (Any method is OK.)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix}$$

$$X = A^{-1}B = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 4+3-8 \\ -2-3+4 \\ 2+3-8 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -3 \end{pmatrix}$$

9. The Gompertz growth model states that if $N(t)$ denotes the population size at time t, then N satisfies the differential equation

$$\frac{dN}{dt} = rN(\ln K - \ln N)$$

where r and K are positive constants, and $N(t) > 0$ for each t.

[7] (a) Find all equilibria of the Gompertz model and use the eigenvalue method to discuss their stability.

$$g(N) = rN(\ln K - \ln N)$$

$$\underbrace{rN}_{>0}(\ln K - \ln N) = 0$$

$$\ln K - \ln N = 0$$

equilibrium $\hat{N} = K$

(since $\hat{N}, K > 0$)

$$g'(N) = r(\ln K - \ln N) + rN\left(-\frac{1}{N}\right) =$$

$$= r(\ln K - \ln N - 1)$$

$$g'(K) = r(\ln K - \ln K - 1) =$$

$$= -r < 0$$

(since $r > 0$ by assumption)

Thus $\hat{N} = K$ is a locally stable equilibrium.

[2] (b) Explain the meaning of the constants r and K.

r ... intrinsic (per capita) growth rate

K ... carrying capacity (optimal population size)

[6] **10.** (a) A simple epidemic model divides the population into three classes: the susceptibles, the infectives and the removed class. Write a system of differential equations that models the spread of the infection.

$$\frac{dS}{dt} = -bSI$$

$$\frac{dI}{dt} = bSI - aI$$

$$\frac{dR}{dt} = aI$$

[2] (b) The model involves two positive constants. Explain how each of them influences the severity of the disease. (You do not need to do any calculations to justify your answer.)

large b $\Rightarrow$ disease more severe
(small b $\Rightarrow$ disease less severe)

large a $\Rightarrow$ disease less severe
(small a $\Rightarrow$ disease more severe)

See HW 8.3.4 part (d)
for explanations (not required here).

[10] **11.** Suppose a crop yield Y depends on nitrogen (N) and phosphorus (P) concentrations as:

$$Y(N,P) = NPe^{-(N+P)}$$

Find the value of (N,P) that maximizes crop yield.

$$Y_N = P\left(e^{-(N+P)} + Ne^{-(N+P)}(-1)\right) =$$
$$= Pe^{-(N+P)}(1-N)$$
$$Y_P = Ne^{-(N+P)}(1-P) \quad \text{by symmetry of } Y$$

critical points: $\begin{cases} Pe^{-(N+P)}(1-N) = 0 \\ Ne^{-(N+P)}(1-P) = 0 \end{cases}$

solve:

$P = 0$ or $N = 0$ not interesting $(Y = 0)$

$e^{-(N+P)} \neq 0$

Hence $(N,P) = (1,1)$ is the only critical point.

$$Y_{NN} = (Y_N)_N = P\left(e^{-(N+P)}(-1)(1-N) + e^{-(N+P)}(-1)\right) =$$
$$= Pe^{-(N+P)}(N-2)$$
$$Y_{PP} = Ne^{-(N+P)}(P-2) \quad \text{by symmetry of } Y$$
$$Y_{NP} = (Y_N)_P = (1-N)\left(e^{-(N+P)} + Pe^{-(N+P)}(-1)\right) =$$
$$= e^{-(N+P)}(1-N)(1-P)$$
$$D = Y_{NN}(1,1) \cdot Y_{PP}(1,1) - \left(Y_{NP}(1,1)\right)^2 =$$
$$= 1 \cdot e^{-2}(-1) \cdot 1 \cdot e^{-2}(-1) - 0^2 = e^{-4} > 0$$

$Y_{NN}(1,1) = -e^{-2} < 0 \Rightarrow$ Yield maximized at $(N,P) = (1,1)$.