Math 154 Midterm 1
Feb. 3, 2006

NAME (printed) : Solution Key
(Family Name) (First Name)

Student Number :

Signature :

(1) Do NOT open this test booklet until told to do so

(2) Do ALL your work in this test booklet

(3) SHOW ALL YOUR WORK

(4) The value of each question is shown below and on the question.

(5) Exam Duration: 50 minutes

(6) NO CALCULATORS

Question	1	2	3	4	5	TOTAL
Score						
Value	10	8	7	8	7	40

1) [Value: 2 points each] Answer the following true (T) or false (F) in the boxes provided. Space is provided for your (optional) comments, should you wish to point out a subtlety to the grader.

$\boxed{F}$ (a) The number 5000 is about 5 orders of magnitude greater than the number 1/200.

$5000/\frac{1}{200} = 1000000 = 10^6 \rightarrow 6$

$\boxed{F}$ (b) If $a_n = 10$ is a fixed point of the recurrence relation $a_{n+1} = f(a_n)$, then $\lim_{n\to\infty} a_n = 10$.

eg $a_{n+1} = 3 - \frac{2}{a_n}$, $a_0 = 10$

1 is a fixed point but $\lim_{n\to\infty} a_n = 2$

$\boxed{T}$ (c) The function $f(x) = \sin(0.2x)$ is periodic and has period 10π.

$\boxed{F}$ (d) The natural logarithm satisfies $\ln x > 0$, for every positive real number x.

eg $\ln \frac{1}{e} = -1$

$\boxed{F}$ (e) For every value of $r > 0$, the Discrete Logistic Recurrence, $x_{n+1} = rx_n(1 - x_n)$, exhibits periodic behaviour for large values of n.

eg it is chaotic for $r > 3.57$

2) [Value: 4+4 points] Here are two problems regarding limits:

(1) It is a fact that if $a_n = \dfrac{\sqrt{n}-1}{\sqrt{n}}$, then $\lim_{n\to\infty} a_n = 1$. Find a value of N such that, if $n > N$ then $|a_n - 1| < 0.01$.

We have $|a_n - 1| = \left| \dfrac{\sqrt{n}-1}{\sqrt{n}} - 1 \right| = \left| \dfrac{-1}{\sqrt{n}} \right| = \dfrac{1}{\sqrt{n}}$

We want that

$$\frac{1}{\sqrt{n}} < 0.01$$

$$\sqrt{n} > 100$$

$$n > 10{,}000$$

Therefore, if $n > \underline{\underline{N = 10{,}000}}$, then $|a_n - 1| < 0.01$.

(2) Calculate the value of $\lim_{n\to\infty} \dfrac{3n^2+2}{4n^2}$. You may use any of the Limit Laws, and you may use the fact $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

$$\lim_{n\to\infty} \frac{3n^2+2}{4n^2} = \lim_{n\to\infty} \frac{n^2\left(3+\frac{2}{n^2}\right)}{4n^2} = \lim_{n\to\infty} \left(\frac{3}{4} + \frac{\frac{2}{n^2}}{4} \right)$$

$$= \lim_{n\to\infty} \frac{3}{4} + \lim_{n\to\infty} \frac{1}{2n^2} \quad \text{(sum law)}$$

$$= \frac{3}{4} + \frac{1}{2}\left(\lim_{n\to\infty} \frac{1}{n}\right) \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right) \quad \text{(product law)}$$

$$= \frac{3}{4} + \frac{1}{2} \cdot 0 \cdot 0 \quad \left(\text{since } \lim_{n\to\infty} \frac{1}{n} = 0\right)$$

$$= 3/4$$

3) [Value: 7 points]
The population of a certain bacterium, *Delirius goddynus*, is growing exponentially with a doubling time of 50 hours. Suppose the population is currently 100. Derive an expression for the approximate population one week from now. (Note: there are 168 hours in a week). Do not attempt to evaluate your expression. You do not have a calculator!

Population at time t is $N(t) = N_0 e^{\lambda t}$

Doubling time is 50 hrs so

$$N(50) = 2N(0)$$

$$N_0 e^{50\lambda} = 2N_0$$

$$e^{50\lambda} = 2$$

$$50\lambda = \ln 2$$

$$\lambda = \frac{\ln 2}{50}$$

Since $N_0 = 100$ we have

$$N(168) = N_0 e^{\lambda(168)}$$

$$= 100 e^{\left(\frac{\ln 2}{50}\right)(168)}$$

$$= 100 \cdot 2^{\frac{168}{50}}$$

4) [Value: 4+2+2 points]
All three parts refer to the sequence $\{a_n\}$ defined by the recurrence

$$a_{n+1} = 3 - \frac{2}{a_n}, \qquad a_0 = 6.$$

(1) Find the fixed points of this recurrence.

We solve $a = 3 - \frac{2}{a}$

$$a^2 - 3a + 2 = 0$$

$$(a-2)(a-1) = 0$$

$\therefore$ $a = 2$ and $a = 1$ are the fixed points

(2) Calculate a_1, a_2, and a_3.

$$a_1 = 3 - \frac{2}{6} = \frac{9}{3} - \frac{1}{3} = \frac{8}{3}$$

$$a_2 = 3 - \frac{2}{8/3} = 3 - \frac{6}{8} = \frac{12}{4} - \frac{3}{4} = \frac{9}{4}$$

$$a_3 = 3 - \frac{2}{9/4} = 3 - \frac{8}{9} = \frac{27}{9} - \frac{8}{9} = \frac{19}{9}$$

(3) The limit $\lim_{n\to\infty} a_n$ exists. What is this limit?

Clearly, this limit is 2 since it is either 1 or 2