Simon Fraser University

MATH 154, Spring 2010

Final Exam

April 22, 2010, 8:30 - 11:30

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Instructions:

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.

2. Fill in the above box.

3. This exam contains 11 pages with a total of 10 questions. Once the exam begins please check to make sure your exam is complete.

4. FOR FULL MARK YOU MUST SHOW ALL YOUR WORK!

5. If you run out of space in a problem, use the space on the back of the previous page and clearly indicate where the solution continues.

6. Only scientific calculators are allowed.

7. No book, paper, or device, other than the usual writing instruments, this booklet and a scientific calculator, shall be within reach of a student during the examination.

8. During the examination, speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.

Question	Marks	Question	Marks
1		6	
2		7	
3		8	
4		9	
5		10	
SubTot.		Total	

1. **[10 marks]** Answer T (true) or F (false) in the boxes provided or leave the box blank. No explanation is necessary. For every correct answer you receive 2, for an incorrect answer you receive -1. You may leave the box empty in which case you receive 0.

a) $\boxed{F}$ A log-log plot of an exponential function $y = 3^x$ is a straight line.

b) $\boxed{F}$ A function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to a^-} f(x) = f(a)$.

c) $\boxed{F}$ If a function $f(x)$ has a local extrema at an interior point a, then $f'(a) = 0$.

d) $\boxed{T}$ Suppose $f(x)$ is twice differentiable on an open interval containing a. If $f'(a) = 0$ and $f''(a) < 0$ then f has a local maximum at $x = a$.

e) $\boxed{F}$ L'Hopital's Rule is the claim: If $f'(x)$ and $g'(x)$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

2. **[10 marks]** Light intensity in lakes decreases exponentially with depth. If $I(z)$ denotes the light intensity at depth z, with $z = 0$ representing the surface, then

$$I(z) = I(0)e^{-\alpha z}, z \geq 0,$$

where α is a positive constant (depends on wavelength of light and water).

a) If we know that 65% of red light is absorbed in the first meter, find α.

100% - 65% = 35% red light intensity at 1 m

$$\frac{I(1)}{I(0)} = 0.35$$

$$e^{-\alpha} = 0.35$$

$$-\alpha = \ln 0.35 \rightarrow \underline{\alpha \approx 1.05}$$

b) Find the depth at which 90% of red light is absorbed.

100% - 90% = 10% red light intensity at z m

$$\frac{I(z)}{I(0)} = 0.10$$

$$e^{-1.05 \cdot z} = 0.1$$

$$-1.05 \cdot z = \ln 0.1 \rightarrow \underline{z \approx 2.19 \text{ m}}$$

The height y (in feet) of a tree as a function of age x in years is given by $y(x) = 132e^{-20/x}$. Can the tree ever reach the height of 200 ft? You must justify your answer for marks.

- $y'(x) = 132 \cdot e^{-\frac{20}{x}} \cdot (-20) \cdot \frac{-1}{x^2} = \frac{264}{x^2} \cdot e^{-\frac{20}{x}} > 0$ for $x > 0$

hence $y(x)$ is increasing on $[0, \infty)$

- $\lim_{x \to \infty} y(x) = \lim_{x \to \infty} 132 \cdot e^{-\frac{20}{x}} = 132 \cdot \lim_{x \to \infty} e^{-\frac{20}{x}} = 132 \cdot 1 = \underline{132}$

- $y(x) \leq 132$ for all $x > 0$ $\rightarrow$ Tree will newer reach 200 ft.

3. **[10 marks]** Calculate the limits

a) $\lim_{x\to\infty} \frac{\sin(x)}{x} =$

- $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ for $x > 0$

- $\lim\limits_{x\to\infty} -\frac{1}{x} = \lim\limits_{x\to\infty} \frac{1}{x} = 0$

- by Sandwitch thm. $\lim\limits_{x\to\infty} \frac{\sin x}{x} = \underline{0};$

b) $\lim_{x\to\infty} \frac{x^4}{e^x} \overset{\frac{\infty}{\infty}}{=\!=} \lim\limits_{x\to\infty} \frac{4x^3}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to\infty} \frac{12x^2}{e^x} \overset{\frac{\infty}{\infty}}{=}$

$= \lim\limits_{x\to\infty} \frac{24x}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to\infty} \frac{24}{e^x} = \underline{0};$

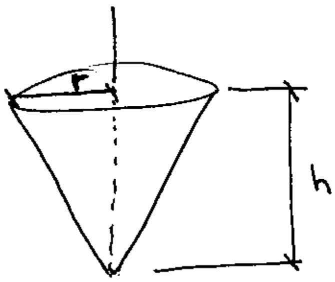
c) $\lim_{x\to 0^+} x^{\sin(x)} =$ y

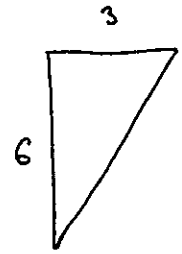
$\ln y = \lim\limits_{x\to 0^+} \sin(x)\cdot\ln(x) = \lim\limits_{x\to 0^+} \frac{\ln(x)}{\csc(x)} \overset{\frac{\infty}{\infty}}{=}$

$= \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\csc(x)\cdot\cot(x)} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to 0^+} \frac{-\sin(x)\cdot\tan(x)}{x} \overset{\frac{0}{0}}{=}$

$= -\lim\limits_{x\to 0^+} \frac{\cos(x)\cdot\tan(x) + \sin(x)\cdot\sec^2(x)}{1} = 0 \rightarrow \underline{y = 1};$

4. **[10 marks]** Suppose we pump water into an inverted right-circular tank at the rate 5 cubic feet per minute. The tank has the height 6 ft and the radius on the top is 3 ft. What is the rate at which the water level is rising when the water is 2 ft deep? You may use the formula for the volume of the right-circular cone of radius r and height h: $V = \frac{1}{3}\pi r^2 h$.





$$\frac{r}{h} = \frac{3}{6} = \frac{1}{2} \Rightarrow r = \frac{h}{2}$$

$$V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$$

$$V(h) = \frac{\pi}{3} \frac{h^2}{4} \cdot h = \frac{\pi \cdot h^3}{12}$$

t – time

• instantaneous rate of change

$$\frac{dV}{dt} = \frac{\pi \cdot 3h^2}{12} \cdot \frac{dh}{dt} = \frac{\pi}{4} \cdot h^2 \cdot \frac{dh}{dt} \qquad (*)$$

• given $h = 2$ and $\frac{dV}{dt} = 5$ by $(*)$ we have

$$5 = \frac{\pi}{4} \cdot (2)^2 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{5}{\pi};$$

The water level is rising at rate $\frac{5}{\pi} \left[\frac{ft}{min}\right]$ when the height is 2 [ft].

5. **[10 marks]** Calculate derivatives of the following functions:

a) Find $f'(x)$ when $f(x) = \sqrt{\sin(2x^2 - 1)}$.

$$f'(x) = \frac{1}{2\sqrt{\sin(2x^2-1)}} \cdot \cos(2x^2-1) \cdot 4x = \frac{2x \cdot \cos(2x^2-1)}{\sqrt{\sin(2x^2-1)}} ;$$

b) Find $\frac{dy}{dx}$ when $xy - y^3 = 1$.

$$\frac{d}{dx}(xy) - \frac{d}{dx} y^3 = \frac{d}{dx} 1$$

$$y + x \cdot \frac{dy}{dx} - 3y^2 \cdot \frac{dy}{dx} = 0 \rightarrow \underline{\frac{dy}{dx} = \frac{y}{3y^2 - x}} ;$$

c) Find $f'(x)$ when $f(x) = \ln(\cos(1 - x))$.

$$f'(x) = \frac{1}{\cos(1-x)} \cdot (-\sin(1-x)) \cdot (-1) = \frac{\sin(1-x)}{\cos(1-x)} =$$

$$= \underline{\tan(1-x)} ;$$

6. **[10 marks]** Find all points on the curve $y = 2x^3 - 4x + 1$ whose tangent line is parallel to the line $y - 2x = 1$.

- line $y - 2x = 1$ has slope 2
- $y' = 6x^2 - 4$ we want $y' = 6x^2 - 4 = 2$

$$x = \pm 1$$

$x_1 = 1 \rightarrow y_1 = 2(1)^3 - 4(1) + 1 = -1$

$x_2 = -1 \rightarrow y_2 = 2(-1)^3 - 4(-1) + 1 = 3$

Points on the curve $y = 2x^3 - 4x + 1$ whose tangent line is parallel to the line $y - 2x = 1$ are $(1, -1)$ and $(-1, 3)$.

7. **[10 marks]**

a) Use the tangent line approximation to approximate the value of ln(1.01).

$$\ln(1.01) \approx L(1.01) = \ln(1) + \ln'(1)(1.01-1) =$$

$$= 0 + 0.01 = \underline{0.01};$$

b) A spherical balloon is being filled with air. When the radius $r = 6$ cm, the radius is increasing at a rate 2 cm/s. How fast is the volume changing at this time?

$$V = \frac{4}{3}\pi r^3 \qquad r = r(t)$$

t - time

$$\frac{dV}{dt} = \frac{4}{3}\pi 3 \cdot r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

when $r = 6$ cm, $\frac{dr}{dt} = 2\ \frac{cm}{s}$. Hence

$$\frac{dV}{dt} = 4\pi \cdot (6)^2 \cdot 2 = \underline{288\pi}; \left[\frac{cm^3}{s}\right]$$

The volume is changing at rate $288\pi\ \frac{cm^3}{s}$ at this time.

8. **[10 marks]**

a) Suppose that $f(x) = x^2 - x - 2$, $x \in [-1, 2]$. Use the Mean Value Theorem to show that there exists point $c \in (-1, 2)$ with horizontal tangent. Find such c.

- $f(x)$ continuous on $[-1, 2]$
- $f(-1) = f(2) = 0$
- by MVT $\exists\, c \in (-1, 2)$ s.t. $f'(c) = \dfrac{f(2) - f(-1)}{2 - (-1)} = 0$
- since $f'(c) = 0$ it follows that the tangent line at $(c, f(c))$ is a horizontal line.

b) Use the Intermediate Value Theorem to conclude that $\sqrt{x^2 + 2} = 2$ has a solution in $(1, 2)$. $f(x) = \sqrt{x^2+2} - 2$.

- $f(x)$ continuous on $[1, 2]$
- $f(1) = \sqrt{1^2 + 2} - 2 = \sqrt{3} - 2 < 0$
- $f(2) = \sqrt{2^2 + 2} - 2 = \sqrt{6} - 2 > 0$
- by IVT $\exists\, c \in (1, 2)$ s.t. $f(c) = 0$

$0 = f(c) = \sqrt{c^2 + 2} - 2 \Rightarrow \sqrt{c^2+2} = 2$

and hence c is a solution.

9. **[10 marks]** Given function $f(x) = xe^x$, $x \in \mathbb{R}$.

a) Find all local/global extrema points (or justify they do not exist).

- $\operatorname{dom} f = \mathbb{R}$
- $f'(x) = e^x + x \cdot e^x = (1+x) \cdot e^x$ for all $x \in \mathbb{R}$
- critical points: $f' = 0 \Rightarrow \underline{x = -1}$

  $f'(x) > 0$ for $x > -1$ $\quad$ $f'(x) < 0$ for $x < -1$
- $x = -1$ - local / global minimum
- $f(x)$ has no local / global max

b) Find all inflection points (or justify they do not exist).

- $f''(x) = e^x + (1+x) e^x$ for all $x \in \mathbb{R}$

  $= (2+x) e^x$
- $f''(x) = 0 \Rightarrow x = -2$ - candidate
- $f''(x) < 0$ for $x < -2$ - f concave down

  $f''(x) > 0$ for $x > -2$ - f concave up
- $x = -2$ is the only inflection point

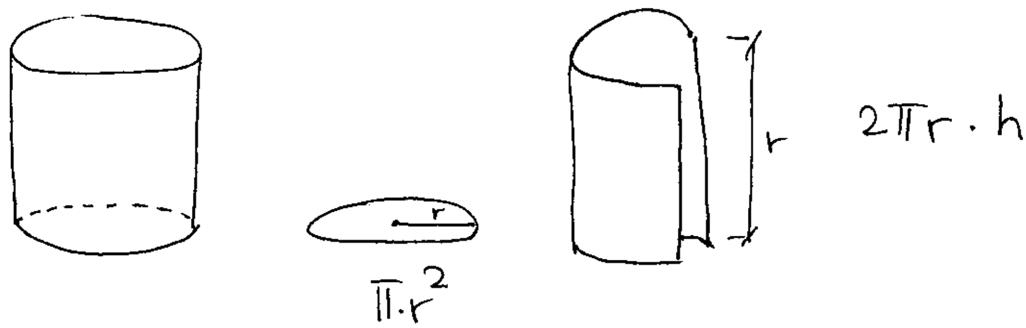
c) Find horizontal asymptotes of $f(x)$ (or justify they do not exist).

$$\lim_{x \to -\infty} x \cdot e^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} (-e^x) = 0$$

x - axis is a horizontal asympt.

$\lim_{x \to \infty} x \cdot e^x = \infty$ - this does not give new asympt.

10. **[10 marks]** A cylindrical can without a top is made to contain $V\ cm^3$ of liquid. Find the dimensions (the radius r of the bottom and the height h) that will minimize the cost of the material to make the can. You may use the fact that the area of circle of radius r is $A = \pi r^2$, and the volume of cylinder of radius r and height h is $V = \pi r^2 h$.



$$V = \pi r^2 \cdot h$$

$$A = \pi \cdot r^2 + 2\pi \cdot r \cdot h$$

$$h = \frac{V}{\pi r^2} > 0$$

$$A(r) = \pi \cdot r^2 + \frac{2V}{r}, \quad r > 0$$

$$A'(r) = 2\pi r + \frac{-2V}{r^2} = \frac{2(\pi r^3 - V)}{r^2}, \quad r > 0$$

- $A'(r) = 0$ when $\pi r^3 - V = 0 \rightarrow r = \sqrt[3]{\frac{V}{\pi}}$

- $A'(r) < 0$ for $0 < r < \sqrt[3]{\frac{V}{\pi}}$
  $A'(r) > 0$ for $r > \sqrt[3]{\frac{V}{\pi}}$

$\Big\}$ $\underline{r = \sqrt[3]{\frac{V}{\pi}}}$; is local / global min.

$$r = \sqrt[3]{\frac{V}{\pi}} \rightarrow h = \frac{V}{\pi} \cdot r^{-2} = \frac{V}{\pi} \cdot \left(\frac{V}{\pi}\right)^{-\frac{2}{3}} = \underline{\sqrt[3]{\frac{V}{\pi}}};$$

Dimensions $r = h = \sqrt[3]{\frac{V}{\pi}}$ minimize area.