STUDENT NUMBER: ____________ NAME: ______________________________

SIMON FRASER UNIVERSITY
DEPARTMENT OF MATHEMATICS
MATH 154

December 15, 2003 FINAL EXAMINATION D. R. Luke

INSTRUCTIONS

1. Do all questions. **This test is 14 pages long.**
2. The questions have several parts. The parts are, mostly, independent. **If you have trouble on an early part of a question, move on to the next part.** Don't assume that you won't be able to get subsequent parts correct.
2. Answer in the space provided. If you need more space use the back of the page.
3. **Show ALL relevant computations.** Marks will be given for complete solutions more than for correct answers. Write statements in complete sentences which clearly indicate your results.
4. No notes or calculators.
5. If doubt exists as to the interpretation of any question, ask for clarification.

FOR INSTRUCTOR'S USE	
Question	Mark
1 [25]	
2 [20]	
3 [70]	
4 [40]	
5 [45]	
Total[200]	

Good luck. Show us what you know!

1. Theorems and Definitions. Fill in the blanks of the following definitions and theorems. Parts $(a) - (e)$ are worth 5 points each.

   (a) Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

   - If ________________ for all $x \in (a, b)$ then f is *strictly increasing* on $[a, b]$.
   - If ________________ for all $x \in (a, b)$ then f is *strictly decreasing* on $[a, b]$.

   (b) The *derivative* of a function f at x, denoted $f'(x)$ is defined by

   $$f'(x) = \underline{\hspace{6cm}}$$

   so long as this limit exists.

   (c) The *Mean Value Theorem* states that, if f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one point $x_* \in (a, b)$ such that

   $$f'(x_*) = \underline{\hspace{6cm}}.$$

   (d) *The Extreme Value Theorem* states that, if f is a continuous on a closed interval $[a, b]$ with $-\infty < a < b < \infty$, then f has a ____________ ____________ and a ____________ ____________ on $[a, b]$.

   (e) *The Intermediate Value Theorem* states that, if f is continuous on the closed interval $[a, b]$ and L is any real number with ____________ or ____________, then there exists at least one number c on the open interval (a, b) such that ____________.

2. Graphical understanding. The following require you to draw *qualitative* sketches of functions illustrating the concepts below. Parts $(a) - (d)$ are worth 5 points each.

   (a) Draw a model for a population that starts with 10 members at time $t = 0$, is increasing at time $t = 1$. On the same graph, draw the *tangent line* approximation to this population at $t = 1$.

   (b) Draw a *monotone decreasing* function that is continuous on $(1, \infty)$ with a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = 1$.

(c) Draw a function that is *convex* (as opposed to concave) and has no global minimum on $\mathbb{R}$.

(d) Draw a function with an inflection point at $x = 1$

3. Analysis. Some of the following questions require short explanations. Please use complete sentences.

   (a) (5pts) Where is the function $f(x) = \ln(x^{3x}) - 4x$ undefined?

   (b) (5pts) On the domain $x \in (0, \infty)$, is the function $h(x) = 3/x$ greater than zero? less than zero? or both? Explain your reasoning.

   (c) (5pts) Solve $3\ln(x) - 1 = 0$ for x.

(d) (5pts) Show that $\lim_{x\to 0^+}(\ln(x^{3x}) - 4x) = 0$.

(e) (5pts) Show that $f'(x) = \frac{d}{dx}\left(\ln(x^{3x}) - 4x\right) = 3\ln(x) - 1$, where it is defined.

(f) (5pts) Show that $f''(x) = \frac{d^2}{dx^2}\left(\ln(x^{3x}) - 4x\right) = 3/x$, where it is defined.

(g) (5pts) Write the equation for the asymptote(s) of the function $g(x) = 3\ln x - 1$.

(h) (5pts) On the domain $x \in (0, \infty)$, is $f(x) = \ln(x^{3x}) - 4x$ convex? concave (down)? or both? How do you know this?

(i) (10pts) Suppose you use Newton's method to find the point $x > 0$ such that $\ln(x^{3x}) - 4x = 0$. If your initial guess is $x_0 = 2$, calculate the next iterate. (You need not simplify your answer, but you must explicitly write out the functions to be evaluated).

(j) (5pts) It turns out that we can solve $\ln(x^{3x}) - 4x = 0$ without the help of a computer. Find x satisfying this equation.

(k) (5pts) On the domain $x \in [0.0001, \infty)$, list the extrema, if these exist, of the function $f(x) = \ln(x^{3x}) - 4x$ and label them according to whether they are points of inflection, local and/or global maxima, or local and/or global minima. Explain how you know that the local extrema are minima and/or maxima.

(l) (10pts) Graph the function $f(x) = \ln(x^{3x}) - 4x$, where it is defined. Specify the points x where f reaches a global maximum and/or minimum, and where f crosses the x-axis.

4. COMPUTATION. Parts $(a) - (d)$ are worth 10 points each.

(a) Calculate

$$\lim_{x \to 0} x^3 \cos\left(\frac{1}{\pi x}\right).$$

(b) Calculate dy/dx if $y = x^{2x^3 \sin(4x+\pi/3)}$.

(c) Evaluate

$$\lim_{x\to 1^+}\left(\frac{1}{7\sin(x-1)}-\frac{1}{7(x-1)}\right).$$

(d) Assume $2x+1>0$. Find the equation of the *normal line* to the curve

$$f(x)=\exp[\ln(4x^2+4x+1)-\ln(2x+1)]-2x+e^2.$$

at $x=1$, that is, find the equation of the line *perpendicular* to f at $x=1$. (*Hint: simplify the expression as much as possible before taking the derivative.*)

5. An application. This problem is about specifying a model for population growth of bacterium under experimental conditions. The growth model we propose is

$$P(t) = P(0)2^{at},$$

that is, the population grows exponentially, base 2, in time. We wish to determine the growth rate a for this species of bacteria from three (3) measurements taken at the same time of day for three (3) consecutive days. We estimate the population by measuring the area of the petri dish that the culture covers. Assuming that the culture grows in radial disks on the petri dishes, we then estimate the population by

$$P(t) = 10^{10} r(t)^2,$$

where $r(t)$ is the radius of the culture at time t.

(a) (10pts) If our measurements for the radius of the culture are only accurate to within 5%, then approximately what relative error, at most, can we expect in our estimates for the population size at time t? In other words, if $\Delta r / r_m \leq .05$, then approximately what is the largest value of $\Delta P / P_m$ (here the subscript $_m$ denotes the quantity derived from direct measurement)?

(b) (5pts) The manufacturer of these bacteria claim that, under optimal conditions, they divide 10 times per day. If you start the experiment with a single cell, what would you expect the population to be at the end of three (3) days under optimal conditions?

(c) (10pts) Assume that population growth is continuous in time, and that, at least initially, the bacteria grow at the optimal, manufacturer-specified rate. Then the growth rate of the population at time $t = 0$ is given by

$$P'(0) = 10P(0).$$

You start the experiment with a single cell, with radius 10^{-5} meters. What is the instantaneous rate of change of the radius of the cell culture at time $t = 0$.

(d) (5pts) The petri dish is started on day $t = 0$ with exactly one (1) bacterial cell. A lab technician measures the radii of the cell cultures, plugs these numbers into the relation above for determining the population from culture radius, and compiles the following data:

$$P(1) = 1.024 \times 10^3; \quad P(2) = 2.62144 \times 10^5; \quad P(3) = 3.3554432 \times 10^7.$$

Your job is to determine the rate of growth of this species of bacteria and to report this model, as well as the data, graphically. What scaling (linear, log-linear, or log-log) should you use to plot the data? What is the corresponding linear model for the data? That is, transform the population model $P(t) = P(0)2^{at}$ into a linear function of time.

(e) (15pts) Next, you find the value of a that yields the *best fit* to the data in the least squares sense, that is, you solve the optimization problem

$$\begin{array}{rl} \text{minimize} & f(a) \\ \text{over} & a \in \mathbb{R} \end{array}$$

where the objective function f is given by

$$f(a) = 14a^2 - 218a + 849.$$

Solve this optimization problem. Use the relevant properties of the objective, i.e. convexity, monotonicity, etc., to justify your answer. Are the conditions in your petri dish "optimal" according to the manufacturer's specification?

(f) (**Bonus:** 5pts) Use your model, with the value you found for a above to predict the **cell population** at the end of the fourth day, i.e. at $t = 4$. (Make sure your answer makes sense.)