Math 152 - Test 2 [4]

#1 a) $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int \tan^3 x \sec^{-1/2} x\,dx$

ALTERNATE SOL'N

$= \int \tan^2 x \sec^{-3/2} x \cdot \sec x \tan x\,dx$

$= \int (\sec^2 x - 1)\sec^{-3/2} x \cdot \sec x \tan x\,dx$ (1)

let $u = \sec x \Rightarrow du = \sec x \tan x\,dx$ (1)

$\therefore \int (u^2-1)u^{-3/2}\,du = \int \left(u^{1/2} - u^{-3/2}\right)du = \frac{2u^{3/2}}{3} + 2u^{-1/2} + C$ (1)

$\therefore \int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \frac{2}{3}\sec^{3/2} x + \frac{2}{\sqrt{\sec x}} + C$ (1) (or equivalent) [deduct 0.5 if "C" is omitted]

---

b) $\int_1^2 \frac{x+1}{x^3+x}\,dx$ [5]

(Partial fractions)

$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ (1)

$\therefore x+1 = A(x^2+1) + (Bx+C)x \quad \therefore A=1,\ B=-1,\ C=1$ (1)

$\int_1^2 \frac{x+1}{x^3+x}\,dx = \int_1^2 \left(\frac{1}{x} - \frac{x-1}{x^2+1}\right)dx = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx$ (1)

(may do this by ~~completing the square or~~ making a trig. substitution)

$= \left[\ln|x| - \frac{1}{2}\ln|x^2+1| + \arctan(x)\right]_1^2$ (1)

$\left[\ln(2) - \frac{1}{2}\ln(5) + \arctan(2)\right] - \left[\ln(1) - \frac{1}{2}\ln(2) + \arctan(1)\right]$

$= \frac{3}{2}\ln(2) - \frac{1}{2}\ln(5) + \arctan(2) - \pi/4$ (1) (or equivalent)

c) $\int_1^2 \frac{1}{\sqrt{2x-x^2}}\,dx$ improper integral - discontinuity at $x=2$ [5]

$$= \lim_{t\to 2^-}\int_1^t \frac{1}{\sqrt{2x-x^2}}\,dx \quad (1)$$

Consider: $\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$ (1) (completing the square)

let $x-1=\sin u$
$dx=\cos u\,du$

$$\int \frac{1}{\sqrt{1-\sin^2 u}}\cdot \cos u\,du = \int du \overset{(1)}{=} u+C$$

$$\therefore \lim_{t\to 2^-}\int_1^t \frac{1}{\sqrt{2x-x^2}}\,dx = \lim_{t\to 2^-}\Big[\sin^{-1}(x-1)\Big]_1^t \quad (1)$$

$$= \lim_{t\to 2^-}\Big[\sin^{-1}(t-1)-\sin^{-1}(0)\Big] = \frac{\pi}{2} \quad (1)$$

$$\therefore \int_1^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \frac{\pi}{2}$$

[if students do not recognise the the integral as improper and do not apply the limit but do the rest correctly - deduct 2.5 marks]

---

d) $\int \frac{\ln x}{x^3}\,dx = \int (\ln x)\cdot x^{-3}\,dx$ (by Parts) [3]

let $u=\ln x$, (1) $dv=x^{-3}\,dx$
$\Rightarrow du=\frac{1}{x}\,dx$, $v=-\frac{x^{-2}}{2}=-\frac{1}{2x^2}$

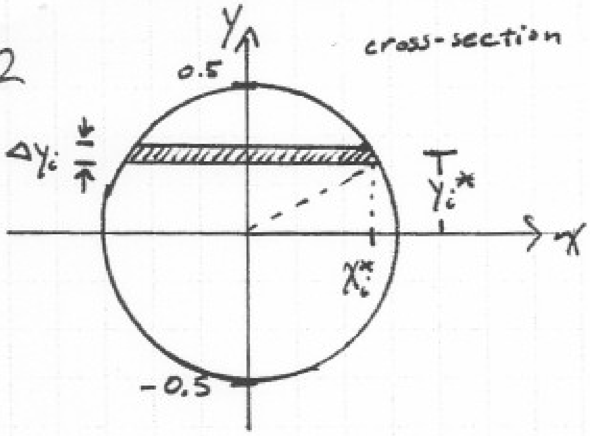
[deduct 0.5 if "C" is omitted]

$$\int \frac{\ln x}{x^3}\,dx = -\frac{\ln x}{2x^2} \;(1)\; -\int -\frac{1}{2x^2}\cdot\frac{1}{x}\,dx$$

$$= \frac{-\ln x}{2x^2}+\frac{1}{2}\int \frac{1}{x^3}\,dx = \frac{-\ln x}{2x^2}+\frac{1}{2}\left[-\frac{1}{2x^2}\right] \overset{(1)}{} + C$$

$$= -\frac{\ln x}{2x^2}-\frac{1}{4x^2}+C \quad \text{(or equivalent)}$$

#2



Consider the ith "slab" of water

Volume: $V_i \approx (2x_i^*)\cdot(\Delta y_i)\cdot(1.5)$

$$x_i^* = \sqrt{(0.5)^2 - (y_i^*)^2}$$

$$= \sqrt{0.25 - (y_i^*)^2}$$

$$\therefore V_i \approx 3\sqrt{0.25 - (y_i^*)^2}\,\Delta y_i \quad 1$$

Force: $F_i = V_i \cdot \rho \qquad (\rho = 6800\ \mathrm{N/m^3})$

$$= 20400\sqrt{0.25 - (y_i^*)^2}\,\Delta y_i \quad 1$$

distance: to top of tank $0.5 - y_i^*$

to tractor $(0.5 - y_i^*) + 1 = d_i$

$\Rightarrow d_i = 1.5 - y_i^*$ 1

$$\therefore W_i = F_i \cdot d_i = \left[20400\sqrt{0.25-(y_i^*)^2}\,\Delta y_i\right]\cdot\left[1.5 - y_i^*\right]$$

$$\therefore W = \int_{-0.5}^{0.5} 20400(1.5-y)\sqrt{0.25-y^2}\,dy \quad 1$$

[Students may not "build" the integral from the ith slab. They must at least justify each component of the integral

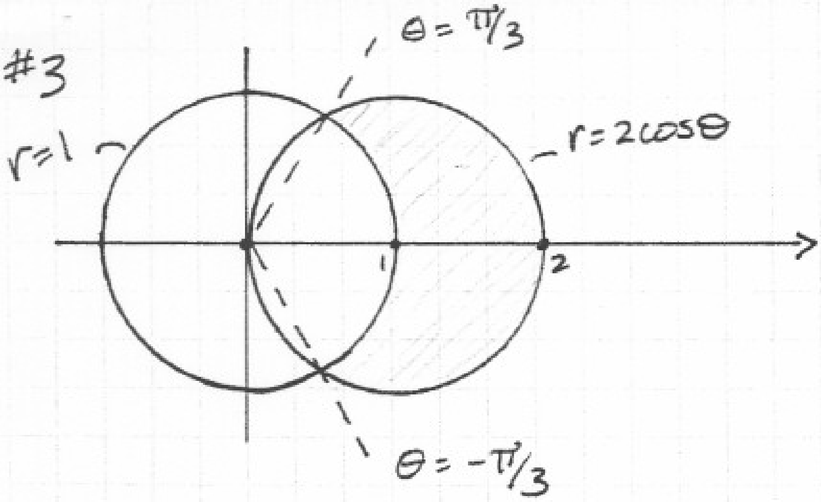
$$\int_{-0.5}^{0.5} 6800\ \overbrace{(1.5-y)}^{\text{distance}}\ \underbrace{3\sqrt{0.25-y^2}\,dy}_{\text{Volume}}$$

1 1 Force 1 1

Math 152

[5]

#3



intersection points:

$$2\cos\theta = 1$$

$$\Rightarrow \theta = \pm \frac{\pi}{3} \quad 1$$

$$A = \int_\alpha^\beta \frac{1}{2}\left(r_o^2 - r_i^2\right) d\theta = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\pi/3} \left[(2\cos\theta)^2 - 1^2\right] d\theta \quad 1$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(4\cos^2\theta - 1\right) d\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[\cancel{4}^{2}\left(\frac{1+\cos 2\theta}{\cancel{2}}\right) - 1\right] d\theta \quad 1$$

$$= \frac{1}{2}\left[\theta + \sin 2\theta\right]_{-\pi/3}^{\pi/3} \quad 1$$

$$= \frac{1}{2}\left[\frac{2\pi}{3} + \sqrt{3}\right] \quad 1$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

#4 $\quad y\sqrt{1-x^2}\cdot\frac{dy}{dx} - x\sqrt{1-y^2} = 0; \quad y(0)=1$ [4]

$$\cancel{\frac{dy}{dx}} \int \frac{y}{\sqrt{1-y^2}}\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx \quad (1)$$

$$\Rightarrow \quad -\sqrt{1-y^2} = -\sqrt{1-x^2} + C \quad (1)$$

$y(0)=1$

$$\Rightarrow \quad 0 = -\sqrt{1-0}+C$$

$$\therefore \quad C = 1 \quad (1)$$

$$\therefore \quad \sqrt{1-y^2} = \sqrt{1-x^2} - 1 \quad (1)$$ (full marks to here)

$$\Rightarrow \quad (1-y^2) = \left(\sqrt{1-x^2}-1\right)^2$$

$$\Rightarrow \quad y^2 = 1 - \left(\sqrt{1-x^2}-1\right)^2$$

$$\therefore \quad y = \sqrt{1-\left(\sqrt{1-x^2}-1\right)^2}$$

(positive, since $y(0)=1$)