Math 152 – Calculus II Second Midterm Solution November 1, 2006

1. Evaluate the following indefinite integrals. Clearly indicate which methods you apply.

   (a) (3 points)

$$\int (\cos x)^5 (\sin x)^3 dx$$

Answer

$$I = \int (\cos x)^5 (\sin x)^3 dx = \int (\cos x)^5 (1 - \cos^2 x) \sin x dx$$

We substitute $u = \cos x$, so that $du = -\sin x dx$ to obtain

$$I = \int -u^5(1-u^2) du = -u^6/6 + u^8/8 + C$$

Substituting back $u = \cos x$ we obtain

$$I = -\frac{1}{6}(\cos x)^6 + \frac{1}{8}(\cos x)^8 + C$$

   (b) (3 points)

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

Answer

We substitute $u = \sqrt{x}$, so that $u^2 = x$ and $2udu = dx$ and obtain

$$\int \frac{e^u}{u} 2udu = \int 2e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$$

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2. Evaluate the following indefinite integrals. Clearly indicate which methods you apply.

   (a) (3 points)

$$\int \frac{x+1}{x^2-2x+2}dx$$

Answer

If we complete the square on the denominator we get $x^2-2x+2=(x-1)^2+1$, so we substitute $u=x-1$, so that $x=u+1$ and $dx=du$. We get

$$\int \frac{u+2}{u^2+1}du = \int \frac{u}{u^2+1}du + 2\frac{1}{u^2+1}du = \frac{1}{2}\ln|u^2+1| + 2\arctan(u) + C$$

Rewriting in terms of x gives:

$$\frac{1}{2}\ln|x^2-2x+2| + 2\arctan(x-1) + C$$

   (b) (3 points)

$$\int \frac{x}{\cos^2(x)}dx$$

Answer

Since $\frac{d}{dx}\tan x = \frac{1}{\cos^2(x)}$ we can write:

$$\int \frac{x}{\cos^2(x)}dx = \int xd(\tan x) = x(\tan x) - \int \tan(x)dx$$

We have

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = \int -\frac{1}{\cos x}d(\cos(x)) = -\ln|\cos x| + C,$$

so the whole answer is:

$$\int \frac{x}{\cos^2(x)}dx = x(\tan x) + \ln|\cos x| + C$$

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3. (3 points) Determine if the following integral converges

$$\int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx$$

Answer

For $x > 0$ we have

$$0 \leq \left(\frac{\sin x}{x}\right)^2 \leq \frac{1}{x^2}.$$

Furthermore,

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{h\to\infty} \int_1^h \frac{1}{x^2} = \lim_{h\to\infty} \left[-\frac{1}{x}\right]_1^h = \lim_{h\to\infty} 1 - \frac{1}{h} = 1$$

Therefore, $\int_1^\infty \frac{1}{x^2} dx$ converges. By the comparison theorem for improper integrals, it follows that the given integral converges as well.

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4. (2 points) Express the arc length of the graph of $\sin(x)$ with $0 \leq x \leq \pi$ as an integral. You don't have to evaluate the integral.

Answer

$$\int_0^\pi \sqrt{1 + (\cos x)^2} dx$$

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5. (2 points) We approximate the value of an integral $\int_a^b f(x)dx$ using the midpoint rule M_n and using Simpson's rule S_n. Suppose we have computed that for $n = 10$ we have

$$\left|\int_a^b f(x)dx - M_{10}\right| \leq 10^{-1} \text{ and } \left|\int_a^b f(x)dx - S_{10}\right| \leq 10^{-1}.$$

   (a) (1 point) How big should we take n to make sure that $\left|\int_a^b f(x)dx - M_n\right| \leq 10^{-5}$? Explain.

Answer

The error bound for the midpoint rule is of the form

$$\left|\int_a^b f(x)dx - M_n\right| \leq C_M/n^2$$

where C_M is some constant depending on the behaviour of $f''(x)$ for $a \leq x \leq b$ but not on n. It is given we can take $C_M/10^2 = 10^{-1}$, so $C_M = 10$.
In order to have $C_M/n^2 \leq 10^{-5}$ we need $n^2 \geq 10^6$, so choosing $n = 1000$ should do the trick.

   (b) (1 point) How big should we take n to make sure that $\left|\int_a^b f(x)dx - S_n\right| \leq 10^{-5}$? Explain.

Answer

The error bound for Simpson's rule is of the form

$$\left|\int_a^b f(x)dx - M_n\right| \leq C_S/n^4$$

where C_S is some constant depending on the behaviour of $f''''(x)$ for $a \leq x \leq b$ but not on n. It is given we can take $C_S/10^4 = 10^{-1}$, so $C_S = 1000$.
Solving $C_M/n^4 \leq 10^{-5}$ shows we should have $n^4 \geq 10^8$, so $n = 100$ should do.

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6. (3 points) Evaluate the following definite integral. Clearly indicate your methods.

$$\int_{-1}^{1} \ln(x^2)dx$$

Answer

Since $\ln(x^2)$ has an asymptote at $x = 0$, this is an improper integral, which we will compute by

$$\int_{-1}^{1} \ln(x^2)dx = \int_{-1}^{0} \ln(x^2)dx + \int_{0}^{1} \ln(x^2)dx.$$

We have

$$\int_{0}^{1} \ln(x^2)dx = \lim_{h\to 0+} \int_{h}^{1} \ln(x^2)dx.$$

By partial integration we have

$$\int_{h}^{1} \ln(x^2)dx = [x\ln(x^2)]_h^1 - \int_h^1 xd(\ln x^2) = (1\cdot 0 - h\ln(h^2)) - \int_h^1 \frac{x}{x^2}\cdot 2xdx = -2h\ln(h) - 2(1-h).$$

In order to compute

$$\lim_{h\to 0+} h\ln(h),$$

we substitute $h = 1/y$ and let $y \to \infty$. We then get

$$\lim_{y\to\infty} \frac{-\ln(y)}{y},$$

which via de l'Hôpital, converges to 0. Therefore,

$$\lim_{h\to 0+} -2h\ln(h) - 2(1-h) = -2$$

and we see that

$$\int_0^1 \ln(x^2)dx = -2.$$

Since $\ln(x^2)$ is an even function, we have

$$\int_{-1}^{-h} \ln(x^2)dx = \int_h^1 \ln(x^2)dx$$

so the other improper integral converges to -2 as well. Adding together yields

$$\int_{-1}^{1} \ln(x^2)dx = -4.$$

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7. (3 points) Evaluate the following definite integral. Clearly indicate your methods.

$$\int_1^3 x\sqrt{1-(x-2)^2}dx$$

Answer

We substitute $u = x - 2$, so that $x = u + 2$ and $du = dx$. Furthermore, if $x = 1$ then $u = -1$ and if $x = 3$ then $u = 1$. Hence, the requested integral equals

$$\int_{-1}^1 (u+2)\sqrt{1-u^2}du = \int_{-1}^1 u\sqrt{1-u^2}du + 2\int_{-1}^1 \sqrt{1-u^2}du$$

The first term is an integral of an odd function over a symmetric interval and hence is 0. The integral in the second term corresponds to the area of half a disc with radius 1, so

$$\int_{-1}^1 \sqrt{1-u^2}du = \pi/2$$

Therefore, the requested integral evaluates to π.
