Solutions Math 152 Midterm 1

Question 1 White & Question 3 Green [6 marks]:

$$I = \int_1^2 \frac{2x^2+x+1}{x}\,dx$$

$$I = \int_1^2 (2x + 1 + \tfrac{1}{x})dx = [x^2 + x + \ln x]_1^2$$

$$= [\{2^2 + 2 + \ln 2\} - \{1^2 + 1 + \ln 1\}] = 4 + \ln 2$$

Answer: $4 + \ln 2$

2 marks for dividing by x & 3 marks for three integrals & 1 mark for the evaluation of limits of integration.

Question 2 White & Question 5 Green [6 marks]:

$$J = \int_{2/\pi}^{3/\pi} \frac{\sin(\frac{1}{t})}{t^2}\,dt$$

Put $x = \frac{1}{t}$ then $dx = -\frac{1}{t^2}dt$

When $t = \frac{2}{\pi}, x = \frac{\pi}{2}$ & when $t = \frac{3}{\pi}, x = \frac{\pi}{3}$. Hence

$J = -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sin x dx = [\cos x]_{\frac{\pi}{2}}^{\frac{\pi}{3}} = \left[\{\cos\frac{\pi}{3}\} - \{\cos\frac{\pi}{2}\}\right] = \frac{1}{2}$

3 marks for reaching $-\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sin x dx$ or equivalent

2 marks for reaching $[\cos x]_{\frac{\pi}{2}}^{\frac{\pi}{3}}$ or equivalent

1 marks for reaching $\frac{1}{2}$

Answer: $\frac{1}{2}$

Question 3 White & Question 4 Green [6 marks]:

$$I = \int_{-3}^{3}\left[x^{101}\cos x + \sqrt{9-x^2}\,\right]dx$$
$$I = J + K = \int_{-3}^{3} x^{101}\cos x dx + \int_{-3}^{3} \sqrt{9-x^2}\,dx$$

$J = 0$ since $x^{101}\cos x$ is an odd function on an interval with midpoint at 0.

$K =$ Area of a semicircle of radius 3 $= \frac{1}{2}(\pi 3^2) = \frac{9\pi}{4}$.Hence

$I = \frac{9\pi}{2}$.

3 marks for J & 3 marks for K

Answer : $\frac{9\pi}{2}$

Question 4 White & Question 7 Green [5 marks]:

If $F(x) = \int_0^{\sin x} 16t^3e^{2t}dt$, *find* $F'\left(\frac{\pi}{6}\right)$.

$F'(x) = \frac{d}{d(\sin x)}\{\int_0^{\sin x} 16t^3e^{2t}dt\,\}\frac{d(\sin x)}{dx} = 16\sin^3 xe^{2\sin x}.\cos x$

(using the Chain Rule & TFTC)

Hence $F'\left(\frac{\pi}{6}\right) = 16(\frac{1}{2})^3(e)(\frac{\sqrt{3}}{2}) = e\sqrt{3}$

4 marks for the product of 4 : $16\sin^3 xe^{2\sin x}.\cos x$(minus 1 mark for each error)

1 marks for $16(\frac{1}{2})^3(e)(\frac{\sqrt{3}}{2})$ or $e\sqrt{3}$

Answer: $e\sqrt{3}$

Question 5 White & Question 1 Green [5 marks]:Approximate

$\int_{\frac{1}{2}}^{\frac{9}{2}} x^2 dx$ by a Riemann sum with a regular partition of four subintervals and with the selection consisting of the midpoint of each subinterval.

$$R_{[P,S]} = \sum_{i=1}^{i=4} f(x_i^*)\Delta x_i \text{ where } x_i^* = \frac{x_{i-1}+x_i}{2}, i = 1,2,3,4.$$

$\Delta x = \frac{\frac{9}{2}-\frac{1}{2}}{4} = 1, \qquad x_0 = \frac{1}{2}, x_1 = \frac{3}{2}, x_2 = \frac{5}{2}, x_3 = \frac{7}{2}, x_4 = \frac{9}{2}.$

$\int_{\frac{1}{2}}^{\frac{9}{2}} x^2 dx \approx (1^2 + 2^2 + 3^2 + 4^2)(1) = 30$

1 mark for $R_{[P,S]} = \sum f(x_i^*)\Delta x_i$ 1 mark for $\Delta x = 1$

1 mark for $x_0 = \frac{1}{2}, x_1 = \frac{3}{2}, x_2 = \frac{5}{2}, x_3 = \frac{7}{2}, x_4 = \frac{9}{2}$ 1 mark for the midpoints

Answer: 30

1 mark for $(1^2 + 2^2 + 3^2 + 4^2)(1) = 30$

Question 6 White & Question 2 Green [6 marks]:

Find the area in the xy-plane bounded by the curves

$y = x^4 - 16$ and $y = 4 - x^2$.

$A = \int_{-2}^{2}[(4 - x^2) - (x^4 - 16)]dx = \left[20x - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2}$

$= \left[\{40 - \frac{8}{3} - \frac{32}{5}\} - \{-40 + \frac{8}{3} + \frac{32}{5}\}\right] = 2(40 - \frac{8}{3} - \frac{32}{5}) = \frac{928}{15}$

2 marks for the limits (including the diagram) 2 marks for the integrand 2 marks for $\{40 - \frac{8}{3} - \frac{32}{5}\} - \{-40 + \frac{8}{3} + \frac{32}{5}\}$ or $\frac{928}{15}$

Answer: $\frac{928}{15}$

Question 7 White & Question 6 Green [6 marks]:

Find the volume of revolution generated by rotating the region $y = \frac{1}{x^2+1}$, $y = 0$, $x = 1$, $x = 2$ in the xy-plane about the y-axis.

Method of Cylindrical shells

$$V = \int_1^2 (2\pi x dx)y = \pi \int_1^2 \frac{2xdx}{1+x^2} = \pi[\ln(1+x^2)]_1^2$$
$$= \pi \ln 2.5$$

2 marks for the limits (including the diagram)

2 marks for the integrand

1 mark for the integration

1 mark for $\pi \ln 2.5$ or equivalent

Method of Washers

$$V = \int_{\frac{1}{5}}^{\frac{1}{2}} \pi(x^2 - 1^2)dy + \int_0^{\frac{1}{5}} \pi(2^2 - 1^2)dy$$
$$= \pi \int_{\frac{1}{5}}^{\frac{1}{2}} (\tfrac{1}{y} - 1 - 1)dy + \frac{3\pi}{5}$$
$$= \pi[\ln y - 2y]_{\frac{1}{5}}^{\frac{1}{2}} + \frac{3\pi}{5} = \pi \ln 2.5$$

2 marks for $\int_{\frac{1}{5}}^{\frac{1}{2}} \pi(x^2 - 1^2)dy$

1mark for $\int_0^{\frac{1}{5}} \pi(2^2 - 1^2)dy$ or equivalent & 1 mark for $\frac{3\pi}{5}$ 1 mark for integrating $\int_{\frac{1}{5}}^{\frac{1}{2}} \pi(x^2 - 1^2)dy$

1 mark for the final answer $\pi \ln 2.5$

Answer: $\pi \ln 2.5$