Math 152 – Calculus II First Midterm Solution October 4, 2006

1. (a) (2 points) Let $a < b$ and let $f(x)$ be a continuous function on $[a, b]$. Give the definition of

$$\int_a^b f(x)dx$$

in terms of a limit of a Riemann sum. Take care to define any auxiliary symbols, like x_i^* and Δx, that you introduce.

Answer

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum i = 1^n f(x_i^*)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$ are sample points.

(b) (2 points) Let $f(x) = (x-1)^3 + 1$. Compute R_4 (the 4 term Riemann sum with sample points taken at the right ends) for $\int_0^2 f(x)dx$.

Answer

If we take the sample points to be the right ends of the intervals, then we take $x_i^* = a + i\Delta x$. We have $a = 0, b = 2, n = 4$, so $\Delta x = \frac{1}{2}$. The values we get are

i	x_i^*	$f(x_i^*)$
1	$\frac{1}{2}$	$\frac{7}{8}$
2	1	1
3	$\frac{3}{2}$	$\frac{9}{8}$
4	2	2

Hence,

$$R_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x = \Delta x \sum_{i=1}^{4} f(x_i^*) = \frac{1}{2}(\frac{7}{8} + 1 + \frac{9}{8} + 2) = \frac{5}{2}$$

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2. (2 points) Let R_n be the n term Riemann sum for $\int_0^2((x-1)^3+1)dx$ with sample points taken at the right end points. Will R_n be larger or smaller than $\int_0^2((x-1)^3+1)dx$? Argue why.

Answer

We will have $R_n \geq \int_0^2((x-1)^3+1)dx$.

Note that the function we are integrating, $f(x)=(x-1)^3+1$, has derivative $f'(x)=3(x-1)^2$, which is a non-negative function. Hence, $f(x)$ is a non-decreasing function.

That means that on any $[a+(i-1)\Delta x, a+i\Delta x]$, we will have that $f(x)$ takes its maximum value at the right end point $\overline{x}_i = a+i\Delta x$. Hence, with the Riemann sum R_n based on the right end points, we will be overestimating the area under the graph of $f(x)$.

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3. (a) (3 points) Compute

$$\int_0^1 x\, e^{-x^2}\, dx$$

Answer

We consider the substitution $u = -x^2$. Then $du = -2xdx$ and if $x = 0$ then $u = 0$ and if $x = 1$ then $u = -1$, so

$$\int_0^1 x\, e^{-x^2} dx = \int_0^{-1} -e^u du = [-\frac{1}{2}e^u]_0^{-1} = -\frac{1}{2}e^{-1} + \frac{1}{2} = \frac{1}{2}(1 - \frac{1}{e})$$

(b) (3 points) Compute

$$\int_{-1}^1 (\sin(x) + \frac{1}{(x-2)^2}) dx$$

Answer

$$\int_{-1}^1 (\sin(x) + \frac{1}{(x-2)^2}) dx = \int_{-1}^1 \sin(x) dx + \int_{-1}^1 \frac{1}{(x-2)^2} dx.$$

The first integral is of an odd function over a symmetric interval so that part vanishes. We are left with

$$\int_{-1}^1 \frac{1}{(x-2)^2} dx = \left[-\frac{1}{x-2} \right]_{-1}^1 = (1 - \frac{1}{3}) = \frac{2}{3}$$

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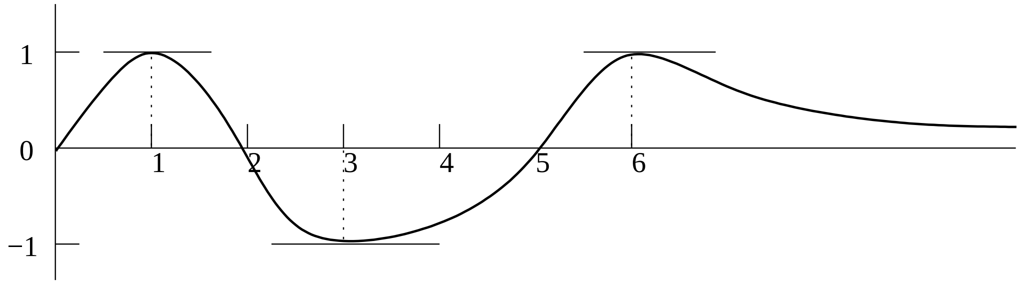
4. (2 points) Consider the function $F(x) = \int_0^x f(t)dt$, where $f(t)$ has the graph below:



For what value of x does $F(x)$ attain its absolute minimum on $[0, 6]$?

Answer

At $x = 5$: The function $F(x)$ measures the net area under the graph of $f(t)$ on the interval $[0, x]$. Hence $F(0) = 0$ and, since $f(x)$ is positive on $[0, 2]$, it increases from there. However, on $[2, 5]$, the function $f(x)$ is negative, so there the area between the graph and the axis counts negative. Hence, on this interval, $F(x)$ is decreasing. Since the area under the axis and above the graph on $[0, 5]$ is bigger than the area under the graph and above the axis on $[0, 2]$, we'll have that $F(5) < 0 = F(0)$. On $[5, \infty]$, $f(x)$ is non-negative and hence $F(x)$ is increasing there. Thus $x = 0$ and $x = 5$ are the only places where $F(x)$ can have a minimum and as argued $F(5) < F(0)$.

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5. Mark each of the following statements **T** (true) or **F** (false) or leave blank: 2 points for a correct answer, 1 for blank, 0 for an incorrect answer. You do not have to justify your answer here.

   (a) (2 points) T/F: $\boxed{\text{F}}$. If $f(x), g(x)$ are continuous functions on $[a, b]$ then

   $$\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right).$$

   (b) (2 points) T/F: $\boxed{\text{T}}$. If $f(x)$ is a continuous function on $[a, b]$ then

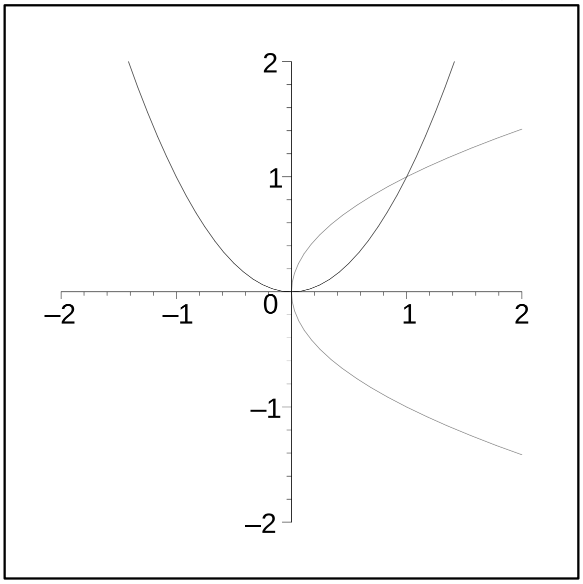
   $$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx.$$

   (c) (2 points) T/F: $\boxed{\text{F}}$. If $f(u)$ and $g(x)$ are continuous functions then

   $$\int_a^b f(g(x))dx = \int_a^b g(u)du.$$

6. Consider the curve $x = y^2$ and $y = x^2$.

   (a) (1 point) Make a sketch of the two curves in in the diagram below.



   (b) (2 points) Write down an integral that computes the area enclosed between the two curves.

Answer

The area that is enclosed lies above $x \in [0, 1]$. The lower curve is described by $y = x^2$ and the upper curve by $y = \sqrt{x}$. Hence, the area can be obtained by computing

$$\int_0^1 (\sqrt{x} - x^2) dx$$

   (c) (2 points) Compute the area enclosed between the two curves.

Answer

We evaluate the integral from the previous part:

$$\int_0^1 (\sqrt{x} - x^2) dx = \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$
