Simon Fraser University

MATH 152
Final Examination

August 4, 2004

Name (*please print*): ANSWER KEY
Last name *Given names*

Student Number: ______________________

Signature: ______________________

NOTES:
- Show all workings. No credit will be given for unsupported answers.
- The use of *any* calculator is strictly prohibited.
- You have 3 hours to complete the examination.
- No notes or aids are permitted during the examination.
- Ensure that your examination contains 8 pages (including this cover page) with 10 questions.

DO NOT WRITE BELOW THIS LINE

Question	Marks	Score
#1	5	
#2	4	
#3	12	
#4	4	
#5	4	
#6	14	
#7	3	
#8	7	
#9	5	
#10	4	
TOTAL	62	

1. Use the <u>definition of the definite integral</u> (with right end-points) to evaluate the integral $\int_0^2 (x^2 - x)\,dx$. [5]

$f(x) = x^2 - x \;; \quad a = 0,\ b = 2$

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$ 1 $\qquad x_i^* = a + i\Delta x = \frac{2i}{n}$ 1

$$\therefore \int_0^2 (x^2 - x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right) \right] \cdot \frac{2}{n} \quad 1$$

$$= \lim_{n\to\infty} \frac{2}{n} \left[\frac{4}{n^2} \sum_{i=1}^{n} i^2 - \frac{2}{n} \sum_{i=1}^{n} i \right]$$

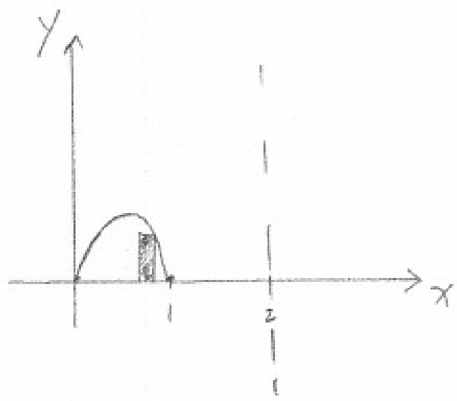
$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) - \frac{4}{n^2} \left(\frac{n(n+1)}{2} \right) \right] \quad 1$$

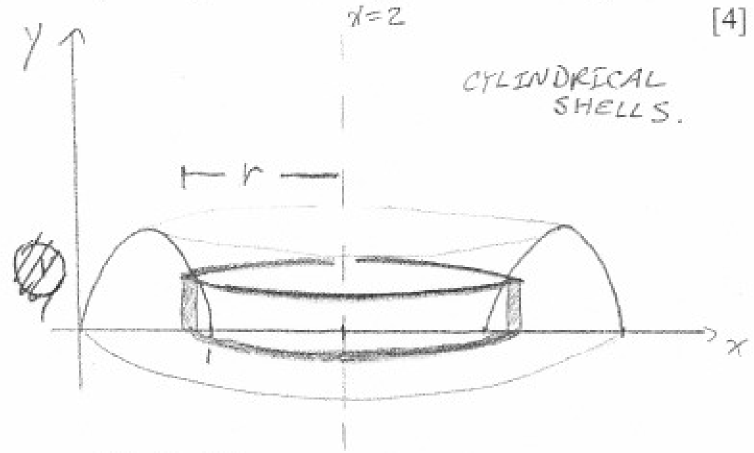
$$= \lim_{n\to\infty} \left[\frac{4}{3} \cdot 1 \cdot \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 2 \cdot 1 \cdot \left(1 + \frac{1}{n}\right) \right]$$

$$= \frac{8}{3} - 2$$

$$= \frac{2}{3} \quad 1$$

2. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$. [4]





$r = 2 - x$

$h = x - x^2$

$$V = \int_a^b 2\pi \cdot r \cdot h\,dx$$

$$\therefore V = \int_0^1 2\pi\,(2 - x)\,(x - x^2)\,dx$$

1 1 1

$$= 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx$$

$$= 2\pi \left[x^2 - x^3 + \frac{x^4}{4} \right]_0^1$$

$$= \frac{\pi}{2} \quad 1$$

3. Integrate the following: partial fractions

a) $\int \frac{x+4}{x^3+x}\,dx$ [4]

$$\frac{x+4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow x+4 = A(x^2+1) + (Bx+C)x$$
$$= (A+B)x^2 + Cx + A$$

$\therefore A=4,\ B=-4,\ C=1$ (1)

$$\int \frac{x+4}{x^3+x}\,dx$$
$$= \int \left(\frac{4}{x} - \frac{4x-1}{x^2+1}\right) dx$$
$$= \int \left(\frac{4}{x} - \frac{4x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$
$$= 4\ln|x| - 2\ln|x^2+1| + \arctan x + C$$
(1) (1) (1)

- Subtract 0.5 if 'C' is omitted
- subtract 0.5 for arithmetic mistakes unless they change the integral significantly.

b) $\int_0^{\pi/2} \sin^4 x \cos^3 x\,dx$ [4]

$$\int_0^{\pi/2} \sin^4 x \cos^3 x\,dx = \int_0^{\pi/2} \sin^4 x \cos^2 x \cdot \cos x\,dx \quad (1)$$
$$= \int_0^{\pi/2} \sin^4 x\,(1-\sin^2 x)\cos x\,dx$$
$$= \int_0^{\pi/2} (\sin^4 x - \sin^6 x)\cos x\,dx \quad (1)$$
$$= \left[\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7}\right]_0^{\pi/2} \quad (1)$$
$$= \frac{1}{5} - \frac{1}{7}$$
$$= \frac{2}{35}. \quad (1)$$

c) $\int e^x \sin(2x)\,dx$ by parts [4]

let $u = e^x$, $dv = \sin(2x)\,dx$ 0.5

$du = e^x dx$, $v = -\frac{\cos(2x)}{2}$

$\therefore \int e^x \sin(2x)\,dx$ 1

$= -\frac{e^x\cos(2x)}{2} + \frac{1}{2}\int e^x \cos(2x)\,dx$

Consider $\int e^x \cos(2x)\,dx$

let $p = e^x$, $dq = \cos(2x)\,dx$

$dp = e^x dx$, $q = \frac{\sin(2x)}{2}$ 0.5

$\therefore \int e^x \cos(2x)\,dx$

$= \frac{e^x \sin(2x)}{2} - \frac{1}{2}\int e^x \sin(2x)\,dx$ 1

$\therefore \int e^x \sin(2x)\,dx$

$= -\frac{e^x\cos(2x)}{2} + \frac{1}{2}\left[\frac{e^x\sin(2x)}{2} - \frac{1}{2}\int e^x \sin(2x)\,dx\right]$

$= -\frac{e^x\cos(2x)}{2} + \frac{1}{4}e^x\sin(2x) - \frac{1}{4}\int e^x\sin(2x)\,dx$

$\therefore \frac{5}{4}\int e^x \sin(2x)\,dx = \frac{e^x}{2}\left(\frac{\sin(2x)}{2} - \cos(2x)\right)$

$\therefore \int e^x \sin(2x)\,dx = \frac{2e^x}{5}\left(\frac{\sin(2x)}{2} - \cos(2x)\right)$ 1

4. Find the length of the curve $y = \frac{2}{3}(x^2+1)^{3/2}$ on $0 \le x \le 2$. [4]

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ 1 $\qquad \frac{dy}{dx} = (x^2+1)^{1/2}\cdot 2x$

$\therefore L = \int_0^2 \sqrt{1 + (x^2+1)\cdot 4x^2}\,dx$

$= \int_0^2 \sqrt{1 + 4x^4 + 4x^2}\,dx$ 1

$= \int_0^2 (2x^2+1)\,dx$ 1

$= \left[\frac{2}{3}x^3 + x\right]_0^2$

$= \frac{22}{3}$ 1

5. Find the equation of a curve in the xy – plane that passes through (0, 3) and whose tangent at any point (x, y) has a slope $\frac{2x}{y}$. (Hint: Set up and solve the differential equation) [4]

1

Slope: $\frac{dy}{dx} = \frac{2x}{y} \Rightarrow \int y\,dy = \int 2x\,dx$

$\therefore \frac{y^2}{2} = x^2 + C$ 1

passes through $(0,3)$: $\frac{3^2}{2} = 0^2 + C \Rightarrow C = \frac{9}{2}$ 1

$\therefore \frac{y^2}{2} = x^2 + \frac{9}{2}$ 1 (full marks to here)

$\Rightarrow y = \sqrt{2x^2+9}$

6. Determine whether the following series converge or diverge. State the test(s) being used.

a) $\sum_{n=1}^{\infty} \frac{n^4}{(1+n^2)^3}$ Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 1 (convergent p-series) [4]

$$\lim_{n\to\infty}\left(\frac{\frac{n^4}{(1+n^2)^3}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{n^6}{(1+n^2)^3} = \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n^2}+1\right)^3}$$ 1

$= 1$ 1

$\therefore \sum_{n=1}^{\infty}\frac{n^4}{(1+n^2)^3}$ 1 converges by the limit comparison test

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OR: $n^6 < (1+n^2)^3$

$\Rightarrow \frac{1}{(1+n^2)^3} < \frac{1}{n^6}$ 1

$\Rightarrow \frac{n^4}{(1+n^2)^3} < \frac{n^4}{n^6} = \frac{1}{n^2}$ 1

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ 1 is a convergent p-series

$\therefore \sum_{n=1}^{\infty}\frac{n^4}{(1+n^2)^3}$ 1 converges by the comparison test (compared to $\sum \frac{1}{n^2}$)

- deduct 0.5 marks if test used is not stated somewhere.

b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ consider $f(x) = \frac{1}{x(\ln x)^2}$, $x \geq 2$ 1

[6]

1
- $f(x) > 0$
- $f(x)$ decreases as x increases
- $f(x)$ is continuous

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx \quad 1$$

$$= \lim_{t\to\infty} \left[\frac{-1}{(\ln x)}\right]_2^t \quad 1$$

$$= \lim_{t\to\infty} \left[-\cancel{\frac{1}{\ln t}}^{\,0} + \frac{1}{\ln 2}\right]$$

$$= \frac{1}{\ln 2} \quad 1$$ (∴ the integral converges)

∴ $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ 1 converges by the integral test.

– deduct 0.5 marks if test used is not stated.

c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)!}{2^{2n-1}}$

[4]

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}(2(n+1)-1)!}{2^{2(n+1)-1}}}{\frac{(-1)^{n+1}(2n-1)!}{2^{2n-1}}} \right| = \lim_{n\to\infty} \left| \frac{(2n+1)!}{2^{2n+1}} \cdot \frac{2^{2n-1}}{(2n-1)!} \right| \quad 1$$

1

$$= \lim_{n\to\infty} \left| \frac{2n\cdot(2n+1)}{2^2} \right|$$

deduct 0.5 marks if test used is not stated

$$= \lim_{n\to\infty} \left| \frac{n(2n+1)}{2} \right| = \infty \quad 1$$

∴ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)!}{2^{2n-1}}$ diverges 1 by the ratio test.

OR: $\frac{(2n-1)!}{2^{2n-1}} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot(2n-1)}{2\cdot 2\cdot 2\cdot \ldots \cdot 2} = \frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot \ldots \cdot\frac{(2n-1)}{2}$ 2

$\frac{2n-1}{2} \geq 1$ for all $n \geq 2$

∴ $\lim_{n\to\infty} \frac{(2n-1)!}{2^{2n-1}} > \frac{1}{2}$ ∴ $\lim_{n\to\infty} \frac{(2n-1)!}{2^{2n-1}} \neq 0$ 1

or some such valid reasoning that shows $\lim_{n\to\infty} a_n \neq 0$

The series diverges by the test for divergence (or the nth-Term Test) 1

7. Calculate the minimum number of terms needed to guarantee that the approximation of the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is correct to two decimal places. [3]

Alternating series, therefore: $|R_n| < a_{n+1}$ (1)

2 decimal places $\Rightarrow a_{n+1} < 0.005$

$$\Rightarrow \frac{1}{n+1} < 0.005 \quad (1)$$

$$\Rightarrow n+1 > 200$$

$$\Rightarrow n > 199$$

$\therefore$ We need at least 200 (1) terms to guarantee the stated accuracy.

8. Determine the interval of convergence for $\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n(n+1)}$. [7]

ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \ (1) = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{(x-2)^n} \right| = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n+2}\right)|x-2|$$

$$= \frac{1}{2}|x-2| \quad (1)$$

series converges if $\frac{1}{2}|x-2| < 1 \Rightarrow |x-2| < 2$

$$\Rightarrow 0 < x < 4 \quad (1)$$

endpoints:

$x=0$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ (1) (Convergent alternating series) (1)

$x=4$: $\sum_{n=0}^{\infty} \frac{(2)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ (1) Compare to harmonic series $\sum_1 \frac{1}{n}$ which diverges.

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

$\therefore \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the limit (1) comparison test.

interval of convergence: $[0, 4)$ $\leftarrow$ deduct 0.5 if not in interval form

9. Use a power series to evaluate the integral $\int_0^1 e^{-x^2}\,dx$. You need only write the first four terms of your answer. [5]

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (1) \qquad \therefore e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$\therefore \int_0^1 e^{-x^2}dx = \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\right)dx \quad (1)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1}\Big|_0^1\right) \quad (1)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \cdot \frac{1}{2n+1}\right)$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \ldots$$

(1) (or equivalent)

Alternate solution.

or students may expand series first:

$$\int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right)dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right]_0^1 \quad (1)$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \ldots \quad (1)$$

(or equivalent)

10. Find a fourth degree Taylor polynomial for $f(x) = \ln x$ about $a = 1$. [4]

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(n)}(a)}{k!}(x-a)^k \quad (1)$$

$$= \frac{\ln(1)}{0!}(x-1)^0 + \frac{(1/1)}{1!}(x-1)^1 + \frac{(-1/1^2)}{2!}(x-1)^2$$

$$+ \frac{(2/1^3)}{3!}(x-1)^3 + \frac{(-6/1^4)}{4!}(x-1)^4 \quad (1)$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$ (1)

$f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = -\frac{6}{x^4}$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \quad (1)$$

(or equivalent)