Simon Fraser University

Math 151-3, Summer 04
Midterm 2

Date: 7 July, 2004
Time: 8:30 am - 9:20 am
Place: C9002

Last Name________________ Given Names______________

Student Number______________

Instructions

1. Do not open this test booklet until instructed to do so.
2. Print your name and write your student number above.
3. No calculators or other calculating devices may be used.
4. Full marks will be awarded for correct, complete and well-organized solutions.
5. You may use the back of any page for rough work.
6. There are 6 pages in this test booklet.

Question	1	2	3	4	5	6	Total
Marks	/12	/6	/6	/6	/5	/10	/45

Good Luck!

1 a) Use L'Hôpital's Rule to find the limit $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$. (6 marks)

Solution $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}\ (\dfrac{0}{0})$

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}\ (\frac{0}{0})$$ (3 marks)

$$=\lim_{x\to 0}\frac{\sin x}{6x}=\frac{1}{6}.$$ (3 marks)

b) Given $y=(\sec x)^x$, find $\dfrac{dy}{dx}$ by the logarithmic differentiation. (6 marks)

Solution If $y=(\sec x)^x$, then $\ln y=\ln(\sec x)^x=x\ln(\sec x)$ (2 marks)

Differentiate both sides of the equation with respect to x,

$(\ln y)'=(x\ln(\sec x))'$.

So,

$$\frac{1}{y}\cdot y'=x'\cdot\ln\sec x+x\cdot(\ln\sec x)'$$ (1 mark)

$$=\ln\sec x+x\cdot\frac{(\sec x)'}{\sec x}=\ln\sec x+x\tan x.$$ (2 marks)

Finally,

$$y'=y\cdot(\ln\sec x+x\tan x)=(\sec x)^x\cdot(\ln\sec x+x\tan x)$$ (1 mark)

2 Use a linear approximation to estimate $\sqrt[3]{26}$. (6 marks)

Solution The linear approximation of $f(x)$ at $x=a$ is

$$f(x) \approx f(a) + f'(a)(x-a)$$

Take $f(x)=\sqrt[3]{x}$ and $a=27$. (2 marks)

$$f'(x)=(\sqrt[3]{x})'=(x^{\frac{1}{3}})'=\frac{1}{3}x^{-\frac{2}{3}}.$$

The linear approximation of $f(x)=\sqrt[3]{x}$ at $x=27$ is

$$\sqrt[3]{x} \approx \sqrt[3]{27}+\frac{1}{3}(27)^{-\frac{2}{3}}\cdot(x-27)=3+\frac{1}{3}\cdot 3^{-2}\cdot(x-27) \quad \text{(2 marks)}$$

So $\sqrt[3]{26} \approx 3+\frac{1}{27}(26-27)=3-\frac{1}{27}=\frac{80}{27}$. (2 marks)

3 Prove that $\ln(1+x)<x$, for $x>0$. (6 marks)

Solution 1 Take $f(x)=\ln(1+x)-x$. f is continuous on $[0,+\infty)$. (2 marks)

Because $f'(x)=[\ln(1+x)-x]'=\frac{1}{1+x}-1<0$, for $x>0$,

f is decreasing on $[0,+\infty)$. (2 marks)

So, $f(0)>f(x)$, for $x>0$, that is

$0=f(0)>\ln(1+x)-x$, or $\ln(1+x)<x$, for $x>0$. (2 marks)

Solution 2 Take $f(x)=\ln(1+x)$.

Since f is continuous on $[0,x]$ and

f is differentiable on $(0,x)$, for $x>0$ (2 marks)

the function satisfies the hypotheses of the mean value theorem.

So, $f(x)-f(0)=f'(c)\cdot(x-0)$, $0<c<x$.

Because $f(0)=0$, $f'(c)=\frac{1}{c+1}<1$ and $x>0$, (2 marks)

$f(x)-0=f'(c)\cdot x<x$, that is

$\ln(1+x)<x$ (2 marks)

4 Suppose that $y = f(x)$ is an implicit function defined by $ae^{-x} + be^{2y} = e^{x-y}$ and $f(0) = 0$ is a local extremum. Determine the numbers a and b.(6 marks)

Solution Since $f(0) = 0$, substituting $x = 0$ and $y = 0$ into $ae^{-x} + be^{2y} = e^{x-y}$,

We have $a + b = 1$. (2 marks)

$f(0) = 0$ is a local minimum, so $f'(0) = 0$. (1 mark)

By the implicit differentiation, we get

$(ae^{-x} + be^{2y})' = (e^{x-y})'$,

$ae^{-x} \cdot (-1) + be^{2y} \cdot 2y' = e^{x-y} \cdot (1 - y')$,

Substituting $x = 0$, $y = 0$ and $y' = 0$ into the equation above,

We get that $-a = 1$, or $a = -1$. (2 marks)

$b = 1 - a = 1 - (-1) = 2$.

So, $a = -1$ and $b = 2$. (1 mark)

5 Find a rational function $y = f(x)$ such that f has a vertical asymptote $x = 1$ and a horizontal asymptote $y = 2$. (5 marks)

Solution Let $f(x) = \dfrac{p(x)}{g(x)}$ be a rational function. (1 mark)

Because $x = 1$ is a vertical asymptote, $x - 1$ is a factor of $g(x)$.

It is convenient to take $g(x) = x - 1$. (2 marks)

Because $y = 2$ is a horizontal asymptote, $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{p(x)}{g(x)} = 2$.

So the degree of $p(x)$ should be the same as that of $g(x)$. (1 mark)

It is convenient to take $p(x) = 2x$.

$f(x) = \dfrac{2x}{x-1}$. (It is not unique) (1 mark)

6 Sketch the graph of $f(x) = e^{-x^2}$. (10 marks)

Note: Locate the $y-$intercept; Determine the interval(s) on which f is increasing or decreasing; Locate any local extremes; Determine the interval(s) on which f is concave upward or downward; Locate any inflection points; Locate any asymptotes and sketch the graph.

Solution a) $f(0) = e^{-0^2} = 1$, so $y-$ intercept is $(0,1)$. (1 mark)

b) $f'(x) = (e^{-x^2})' = e^{-x^2}(-2x)$.

Let $f'(x) = 0$, we get the critical number $x = 0$.

Interval	$(-\infty, 0)$	0	$(0, +\infty)$
k	-1		1
$f'(k)$	$+$		$-$
$f'(x)$	$+$		$-$
$f(x)$	↗	Local max.	↘

So, f is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$.

$f(0) = 1$ is a local maximum. (3 marks)

c) $f''(x) = [e^{-x^2}(-2x)]' = e^{-x^2}(-2x)^2 + e^{-x^2}(-2) = (4x^2 - 2)e^{-x^2}$.

Let $f''(x) = 0$, we get $x_1 = -\frac{\sqrt{2}}{2}$ and $x_2 = \frac{\sqrt{2}}{2}$.

Interval	$(-\infty, -\frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}$	$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}$	$(\frac{\sqrt{2}}{2}, +\infty)$
k	-1		0		1
$f''(k)$	$+$		$-$		$+$
$f''(x)$	$+$	**Inflection point**	$-$	**Inflection point**	$+$
$f(x)$	$\cup$		$\cap$		$\cup$

So, f is concave upward on $(-\infty, -\frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2}, +\infty)$.

f is concave downward on $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

$(-\frac{\sqrt{2}}{2}, e^{-\frac{1}{2}})$ and $(+\frac{\sqrt{2}}{2}, e^{-\frac{1}{2}})$ are inflection points. (3 marks)

d) Because $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} e^{-x^2} = 0$,

$y = 0$ is a horizontal asymptote. (1 mark)

e) Sketch the graph. (2 marks)