Math 151 Section D1
Midterm 2 Fall 2006
Solutions - Version 1

1. [**3 marks**] Mark each statement **T** (True) or **F** (False):

T If f is differentiable then $\frac{d}{dx}f(\sqrt{x}) = \frac{f'(\sqrt{x})}{2\sqrt{x}}$.

F $\frac{d}{dx}(\ln 10) = \frac{1}{10}$

T $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = \frac{2}{3}$

T The 99th derivative of $y = \sinh x$ is $y = \cosh x$.

F If $f(x)$ has an inverse function, $g(x)$, then the derivative of $g(x)$ is $1/f'(x)$.

F If $f'(c) = 0$, then f has a local maximum or minimum at c.

Marking: Each correct answer **0.5 marks**.

2. (a) [**3**] Use implicit differentiation to find the slope of the tangent line to the curve $\sin(xy) = y$ at the point $(\pi/2, 1)$.

Solution: To find the slope we need to calculate $\frac{dy}{dx}\Big|_{(x,y)=(\pi/2,1)}$.
From
$$(\cos(xy))(y + xy') = y'$$
for $x = \pi/2$ and $y = 1$ we get
$$\cos(\frac{\pi}{2} \cdot 1) \cdot (1 + \frac{\pi}{2}y') = y'.$$
It follows that the slope of the tangent line to the curve $\sin(xy) = y$ at the point $(\pi/2, 1)$ equals
$$\frac{dy}{dx}\Big|_{(x,y)=(\pi/2,1)} = 0.$$
Marking:

- Correct implicit differentiation: **1.5 marks**.
- Plugging the correct values for x and y: **0.5 marks**.
- Correct/consistent slope: **1 mark**.

(b) [**3**] Find y" if $y = x^{2x}$, $x > 0$.

Solution: To find y' we take logarithms of both sides and differentiate:
$$\begin{aligned} \ln y &= 2x \ln x \\ \frac{y'}{y} &= 2\ln x + 2x \cdot \frac{1}{x} \\ y' &= y \cdot (2\ln x + 2) = 2 \cdot x^{2x} \cdot \ln(ex) \end{aligned}$$
For y'' we differentiate $y' = y(2\ln x + 2) = 2y\ln(ex)$:
$$y'' = 2y' \ln(ex) + \frac{2y}{x} = 4 \cdot x^{2x} \cdot (\ln ex)^2 + 2x^{2x-1}$$
Marking:

- Attempt to use logarithms: **0.5 marks**.
- Correct y': **1.5 marks**.
- Correct/consistent y'': **1 mark**.

3. [**4**] Show that $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$.

Solution: We start with the fact that

$$y = \tanh^{-1} x \Leftrightarrow \tanh y = x.$$

It follows that

$$\begin{aligned}
x &= \frac{\sinh y}{\cosh y} \\
x &= \frac{e^y - e^{-y}}{e^y + e^{-y}} \\
x &= \frac{e^{2y} - 1}{e^{2y} + 1} \\
(e^{2y} + 1)x &= e^{2y} - 1 \\
(x - 1)e^{2y} &= -1 - x \\
e^{2y} &= \frac{1 + x}{1 - x} \\
2y &= \ln\left(\frac{1 + x}{1 - x}\right) \\
y &= \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)
\end{aligned}$$

We note that $e^{2y} = \frac{1+x}{1-x} > 0$ implies that $-1 < x < 1$.

Marking:

- Stating the initial equivalence: **1 mark**.
- Solving for y: **2 marks**.
- Stating and justifying the domain of $\tanh x$: **1 mark**.

4. [**6**] Two ships depart from the same small island at the same time. One ship, the *Queen Algebra*, is going east of the island and the other ship, the *Queen Geometry*, is going north of the island. At a certain time the *Queen Algebra* is sailing at 35 km/h and is 30 km from the island and the *Queen Geometry* is sailing at 45 km/h and is 40 km from the island. At what rate is the distance between them increasing at that time?

   Solution: Let the point I represent the island, let the point A represent the position of the *Queen Algebra* t hours after the two ships departed from the island, and let the point G represent the position of the *Queen Geometry* at the time t. The triangle AGI is with the right angle at the vertex I. Let $x = x(t)$ be the distance between the points A and I, let $y = y(t)$ be the distance between the points G and I, and let $z = z(t)$ be the distance between the points A and G. Thus, z represents the distance between the *Queen Algebra* and the *Queen Geometry* at time t. The question is to evaluate $\frac{dz}{dt}$ when $x = 30$ and $y = 40$.

   First we note that $z^2 = x^2 + y^2$ and that this implies that

   $$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

   At the time when $x = 30$ and $y = 40$, by the Pythagorean Theorem, we get that

   $$z = \sqrt{30^2 + 40^2} = \sqrt{2500} = 50.$$

   Thus, $50\frac{dz}{dt} = 30\frac{dx}{dt} + 40\frac{dy}{dt} = 30 \cdot 35 + 40 \cdot 45 = 2850$. Therefore the distance between the two ships is increasing 57 km/h at the time when the *Queen Algebra* is 30 km from the island and the *Queen Geometry* is 40 km from the island.

   Marking:

   - Stating and justifying the relationship among the three distances: **2 marks**.
   - Correct implicit differentiation y: **1.5 marks**.
   - Correct value of the distance between the two ships at the time of interest: **0.5 marks**.
   - Correct/consistent value of the rate at the time of interest: **2 marks**.

5. [5] **Happy Halloween!** When a murder is committed, the body, originally at 37°C, cools according to Newton's Law of Cooling. Suppose that after two hours the temperature is 35°C, and that the temperature of the surrounding air is constant at 20°C. Find the temperature, H, of the body as a function of t, the time in hours since the murder was committed.

   Note: Leave your answer in the exact form, i.e., as an expression that contains logarithms.

   Solution: Newton's Law of Cooling states that the temperature H of the body at time t is given by $\frac{dH}{dt} = k(H - T_s)$ what is the same as $H = T_s + (H_0 - T_s)e^k t$ where T_s is the temperature of the surroundings, H_0 is the initial temperature of the body, and k is a constant.

   It is given that $H_0 = H(0) = 37$, $H(2) = 35$, and $T_s = 20$. From $35 = 20 + (37 - 20)e^{2k}$ we get that:

$$\begin{aligned} 35 &= 20 + 17e^{2k} \\ e^{2k} &= \frac{15}{17} \\ 2k &= \ln\frac{15}{17} \\ k &= \frac{1}{2} \cdot \ln\frac{15}{17} \end{aligned}$$

   Therefore the temperature, H, of the body as a function of t, the time in hours since the murder was committed is given by

$$H = 20 + 15e^{\frac{t}{2} \cdot \ln\frac{15}{17}}.$$

   Marking:

   - Stating Newton's Law of cooling and correctly entering the given values: **2 marks**.
   - Evaluating k: **2 marks**.
   - Correct/consistent function H: **1 mark**.

6. (a) **[2]** State the Extreme Value Theorem.

Solution: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Marking: All or nothing.

(b) **[4]** Find the absolute maximum and minimum values of the function

$$f(x) = 5x^{2/3} - x^{5/3}$$

on the interval $[-1, 1]$.

Solution: First we find the critical numbers. From

$$\begin{aligned} f'(x) &= \frac{10}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{\frac{2}{3}} \\ &= \frac{5}{3} \cdot \left(\frac{2}{\sqrt[3]{x}} - \sqrt[3]{x^2} \right) \\ &= \frac{5(2-x)}{3\sqrt[3]{x}} \end{aligned}$$

we conclude that $f'(2) = 0$ and that the first derivative of f is not defined at $x = 0$. Thus the critical points are $x = 0$ and $x = 2$. Since only $x = 0$ belongs to the interval $[-1, 1]$, we calculate $f(0) = 0$.

To find the absolute maximum and minimum of f on $[-1, 1]$, we need to calculate the values of f at the endpoints of the interval: $f(-1) = 5 + 1 = 6$ and $f(1) = 5 - 1 = 4$.

Therefore on $[-1, 1]$, f attains the absolute maximum value $f(-1) = 6$ and the absolute minimum value $f(0) = 0$.

Marking:

- Correct first derivative: **1 mark**
- Correct/consistent critical numbers **in the range**: **1 mark**
- Evaluating values of at the critical numbers and the end points: **1 mark**
- Correct/consistent absolute max/min: **1 mark** (If a student got the absolute max/min out of the range: **0 marks**.)