SOLUTIONS

Q1.
Evaluate the following limits, if they exist. You must provide reasoning which **cannot** be simply "this is what the calculator says."

(a) [2 marks] $\lim_{x \to -3^+} \frac{|x+3|}{x^2-9} = \lim_{x \to -3^+} \frac{|x+3|}{(x+3)(x-3)}$

$= \lim_{x \to -3^+} \frac{1}{x-3}$, since $x+3>0$ when $x>-3$

$= -\frac{1}{6}$

(b) [2 marks] $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} =$

$\lim_{\theta \to 0} \frac{\sin\theta / \cos\theta}{\theta}$

$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}$

$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta}$

$= 1 \cdot 1 = 1$

Q1. (continued)

(c) [2 marks] $\lim_{x\to\infty} \frac{5x^2+26-3x}{42+10x^2} =$

$$= \lim_{x\to\infty} \frac{5 + {}^{26}/_{x^2} - {}^{3}/_{x}}{{}^{42}/_{x^2} + 10}$$

(DIVIDING BOTH NUMERATOR & DENOMINATOR BY x^2)

$$= \frac{\lim_{x\to\infty}\left(5 + \frac{26}{x^2} - \frac{3}{x}\right)}{\lim_{x\to\infty}\left(\frac{42}{x^2} + 10\right)}$$

$$= \frac{\lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{26}{x^2} - \lim_{x\to\infty} \frac{3}{x}}{\lim_{x\to\infty} \frac{42}{x^2} + \lim_{x\to\infty} 10}$$

$$= \frac{5 + 0 - 0}{0 + 10} = \frac{5}{10} = \frac{1}{2}$$

Q2.

(a) [1 mark] State the definition of continuity for a function $F(x)$ at a number a.

IF

$$\lim_{x \to a} F(x) = F(a)$$

Suppose that

$$g(x) = \begin{cases} x - 2 & \text{if } x < 1; \\ x^2 + 1 & \text{if } 1 \leq x < 3; \\ k/x & \text{if } 3 \leq x, \end{cases}$$

where k is some constant.

(b) [2 marks] Is the function g continuous at $x = 1$? Justify your answer.

No.

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (x-2) = 1-2 = -1$$

AND $$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (x^2+1) = 1^2+1 = 2$$

SINCE $\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x)$, WE CONCLUDE THAT $\lim_{x \to 1} g(x)$ DOESN'T EXIST. SO g CANNOT BE CONTINUOUS AT $x = 1$.

Q2 continues on the next page.

Q2. (continued)

(c) [1 mark] True or false? Since $g(0) = -2$ and $g(1) = 2$, by the Intermediate Value Theorem the function $g(x)$ must have a root in the interval $(0, 1)$. Justify your answer.

FALSE. TO USE THE INTERMEDIATE VALUE THEOREM, WE REQUIRE g TO BE CONTINUOUS ON THE CLOSED INTERVAL $[0,1]$. BUT FROM PART (b), g IS NOT CONTINUOUS AT $x=1$ WHICH IS IN THAT INTERVAL.

IN FACT, THE FUNCTION g IS NEGATIVE ON THE INTERVAL $[0,1)$ AND THEN JUMPS TO A POSITIVE VALUE AT $x=1$.

(d) [2 marks] Find the constant k that makes the above function $g(x)$ continuous at $x = 3$. Then show that g is continuous at $x = 3$.

WE WANT $\lim_{x\to 3} g(x) = g(3)$.

FOR $\lim_{x\to 3} g(x)$ TO EXIST, WE REQUIRE $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^+} g(x)$

NOW, $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} (x^2+1) = 9+1 = 10$

AND $\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} \frac{k}{x} = \frac{k}{3}$

THUS WE WANT $\frac{k}{3} = 10$, WHICH MEANS $k = 30$.

WITH $k=30$, $\lim_{x\to 3} g(x) = 10$. SINCE $g(3) = \frac{k}{3} = 10$,

WE HAVE $\lim_{x\to 3} g(x) = g(3)$ AND SO g IS CONTINUOUS AT $x=3$.

Q3.
(a) [1 mark] Give the definition of the derivative of a function f.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{OR} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

(b) [4 marks] Using the definition from part (a), evaluate $f'(3)$ for the function

$$f(x) = \sqrt{x+1}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} \right)$$

$$= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x+1}}$$

So, $f'(3) = \dfrac{1}{2\sqrt{3+1}} = \dfrac{1}{4}$

OR

$$f'(3) = \lim_{x\to 3} \frac{\sqrt{x+1} - \sqrt{3+1}}{x-3}$$

$$= \lim_{x\to 3} \frac{\sqrt{x+1} - 2}{x-3}$$

$$= \lim_{x\to 3} \left(\frac{\sqrt{x+1} - 2}{x-3} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} \right)$$

$$= \lim_{x\to 3} \frac{(x+1) - 2^2}{(x-3)\left(\sqrt{x+1} + 2\right)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)\left(\sqrt{x+1} + 2\right)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1} + 2}$$

$$= \frac{1}{\sqrt{3+1} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

Q4.
A particle is moving in a straight line. It's position function at time t seconds is given by

$$s(t) \;=\; \tfrac{1}{4}t^4 - 4.5t^2 + 2\,, \text{ where } t \geq 0.$$

(a) [3 marks] Using any method you like, find an expression for the velocity of the particle at time t.

VELOCITY $= \frac{ds}{dt}$, AND SO

$$v(t) = \frac{d}{dt}\left(\frac{1}{4}t^4 - 4.5t^2 + 2\right) = \frac{d}{dt}\left(\frac{1}{4}t^4\right) - \frac{d}{dt}\left(4.5t^2\right) + \frac{d}{dt}(2)$$

$$= t^3 - 9t + 0$$

$$= t^3 - 9t$$

(b) [2 marks] Find the times t at which the particle is not moving.

THE PARTICLE IS NOT MOVING WHEN $v(t) = 0$

THAT IS, $t^3 - 9t = 0$

OR, $t(t^2 - 9) = 0$

WHICH IS, $t(t-3)(t+3) = 0$

SINCE $t \geqslant 0$, WE HAVE TWO SOLUTIONS, $t = 0$ AND $t = 3$

SO THE PARTICLE IS NOT MOVING AT $t = 0$ SECONDS AND AT $t = 3$ SECONDS

Q5.
Evaluate the following.

(a) [2 marks] y' , if $y = 4x^{1/2} - 7\cos x$

$$y' = 4 \cdot \frac{1}{2} x^{-1/2} - 7(-\sin x)$$

$$= 2x^{-1/2} + 7\sin x$$

(b) [2 marks] $\dfrac{df}{dx}$, if $f(x) = \sin(x^3 + 2)$. LET $f(u) = \sin u$

AND $u = g(x) = x^3 + 2$.

THEN, $$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{du}(\sin u) \cdot \frac{d}{dx}(x^3+2)$$

$$= \cos u \cdot (3x^2)$$

$$= \cos(x^3+2) \cdot (3x^2)$$

$$= 3x^2 \cos(x^3+2)$$

Q5. (continued)

(c) [2 marks] $\frac{d}{dt}\left(\sqrt[3]{t}\, e^t\right) =$

$$= \sqrt[3]{t} \cdot \frac{d}{dt}(e^t) + e^t \cdot \frac{d}{dt}(\sqrt[3]{t})$$

$$= \sqrt[3]{t} \cdot e^t + e^t \cdot \left(\frac{1}{3} t^{-2/3}\right)$$

SINCE
$\sqrt[3]{t} = t^{1/3}$

$$= e^t \sqrt[3]{t} + \frac{e^t}{3t^{2/3}}$$

$$= e^t \left[\sqrt[3]{t} + \frac{1}{3\sqrt[3]{t^2}}\right]$$