Simon Fraser University

Math 151-3, Summer 04
Midterm 1

Date: 2 June, 2004
Time: 8:30 am - 9:20 am
Place: C9002

Last Name________________ Given Names______________

Student Number______________

Instructions

1. Do not open this test booklet until instructed to do so.
2. Print your name and write your student number above.
3. No calculators or other calculating devices may be used.
4. Full marks will be awarded for correct, complete and well-organized solutions.
5. You may use the back of any page for rough work.
6. There are 6 pages in this test booklet.

Question	1	2	3	4	5	6	Total
Marks	/10	/9	/4	/6	/10	/6	/45

Good Luck!

1. a) Find the limit $\lim_{x\to 0}\frac{x}{4(\sqrt{x+1}-1)}$. (4 marks)

Solution $\lim_{x\to 0}\frac{x}{4(\sqrt{x+1}-1)}=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{4(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$ (2 marks)

$=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{4[(\sqrt{x+1})^2-1]}=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{4(x+1-1)}=\lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{4x}$ (1 mark)

$=\lim_{x\to 0}\frac{\sqrt{x+1}+1}{4}=\frac{1+1}{4}=\frac{1}{2}$ (1 mark)

b) Find all numbers c and d so that f defined by

$$f(x)=\begin{cases}\frac{\sin cx}{x}, x<0\\ 4, x=0\\ x+d^2, x\geq 0\end{cases}$$

will be continuous at $x=0$. (6 marks)

Solution $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\frac{\sin cx}{x}=\lim_{x\to 0^-}\frac{c\sin cx}{cx}=c$ (1 mark)

$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(x+d^2)=d^2$ (1 mark)

$f(0)=4$ (1 mark)

If the function is continuous at $x=0$, then $\lim_{x\to 0}f(x)=f(0)$

that is, $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}f(x)=f(0)$. (1 mark)

Thus, we get $c=d^2=4$.

Finally, $c=4$ and $d=2$ or $d=-2$ (2 marks)

2. Find y'

a) $y = \dfrac{2x^2 - \sqrt[3]{x}}{\sqrt{x}}$ (4 marks)

Solution 1 $y = 2x^{2-\frac{1}{2}} - x^{\frac{1}{3}-\frac{1}{2}} = 2x^{\frac{3}{2}} - x^{-\frac{1}{6}}$ (2 marks)

$$y' = (2x^{\frac{3}{2}} - x^{-\frac{1}{6}})' = 2(x^{\frac{3}{2}})' - (x^{-\frac{1}{6}})' = 3x^{\frac{1}{2}} + \frac{1}{6}x^{-\frac{7}{6}}$$ (2 marks)

Solution 2 $y' = \left(\dfrac{2x^2 - \sqrt[3]{x}}{\sqrt{x}}\right)' = \dfrac{(2x^2 - \sqrt[3]{x})' \cdot \sqrt{x} - (2x^2 - \sqrt[3]{x}) \cdot (\sqrt{x})'}{(\sqrt{x})^2}$ (2 marks)

$$= \frac{(4x - \frac{1}{3}x^{-\frac{2}{3}}) \cdot \sqrt{x} - (2x^2 - \sqrt[3]{x}) \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{4x^{\frac{3}{2}} - \frac{1}{3}x^{-\frac{1}{6}} - x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{6}}}{x}$$

$$= \frac{3x^{\frac{3}{2}} + \frac{1}{6}x^{-\frac{1}{6}}}{x} = 3x^{\frac{1}{2}} + \frac{1}{6}x^{-\frac{7}{6}}$$ (2 marks)

b) $y = (1 + \sqrt{x^2 + 1})^{100}$ (5 marks)

Solution $y' = [(1 + \sqrt{x^2 + 1})^{100}]' = 100(1 + \sqrt{x^2 + 1})^{99}(1 + \sqrt{x^2 + 1})'$ (2 marks)

$$= 100(1 + \sqrt{x^2 + 1})^{99}[1' + (\sqrt{x^2 + 1})'] = 100(1 + \sqrt{x^2 + 1})^{99}[(x^2 + 1)^{\frac{1}{2}}]'$$ (1 mark)

$$= 100(1 + \sqrt{x^2 + 1})^{99} \cdot \frac{1}{2}(x^2 + 1)^{\frac{1}{2}-1}(x^2 + 1)'$$ (1 mark)

$$= 100(1 + \sqrt{x^2 + 1})^{99} \cdot \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} 2x = \frac{100(1 + \sqrt{x^2 + 1})^{99} x}{\sqrt{x^2 + 1}}$$ (1 mark)

3. Given $f(x) = (x+1)(x+2)\cdots(x+100)$, find $f'(-1)$. (4 marks)

Solution $f'(x) = [(x+1)(x+2)\cdots(x+100)]'$

$= (x+1)'(x+2)\cdots(x+100) + (x+1)(x+2)'\cdots(x+100) + \cdots + (x+1)(x+2)\cdots(x+100)'$

(2 marks)

$= (x+2)(x+3)\cdots(x+100) + (x+1)(x+3)\cdots(x+100) + \cdots + (x+1)(x+2)\cdots(x+99)$

(1 mark)

$f'(-1) = (-1+2)\cdots(-1+100) + 0 + \cdots + 0 = 1\cdot 2\cdots 99 = 99!$ (1 mark)

4. Apply the intermediate value property of continuous functions to show that the equation $x^3 - 3x^2 + 1 = 0$ has a solution on $[0,1]$. (6 marks)

Solution Take $f(x) = x^3 - 3x^2 + 1$. (1 mark)

$f(0) = 0^3 - 3\cdot 0^2 + 1 = 1 > 0$ (1.5 marks)

$f(1) = 1^3 - 3\cdot 1^2 + 1 = -1 < 0$ (1.5 marks)

Since f is continuous on $[0,1]$ (1 mark)

and $f(0)\cdot f(1) < 0$, by the intermediate value property, there exists c on $[0,1]$ such that $f(c) = 0$. (1 mark)

5. a) Find the equation of the tangent to the curve $y = x^2$ that is perpendicular to the line $y = \frac{1}{2}x + 1$. (6 marks)

Solution The slope of the tangent to the curve $y = x^2$ is

$y' = (x^2)' = 2x$ (1 mark)

Since the tangent line is perpendicular to the line $y = \frac{1}{2}x + 1$,

We have that $2x \cdot \frac{1}{2} = -1$, that is $x = -1$. (2 marks)

Substituting $x = -1$ into the function $y = x^2$, we get $y = (-1)^2 = 1$. (1 mark)

So the slope of the tangent is $y'\big|_{x=-1} = 2x\big|_{x=-1} = -2$ and the point of tangency is $(-1,1)$.

The equation of the tangent is

$y - 1 = -2(x+1)$, that is $y = -2x - 1$. (2 marks)

b) The area of a circle is decreasing at the rate of $2\pi\ cm^2/s$. At what rate is the radius of the circle decreasing when its area is $25\pi\ cm^2$? (4 marks)

Solution Let r be the radius of a circle, A the area of the circle.

So $A = \pi r^2$. (1 mark)

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = (\pi r^2)'_r \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \qquad (1) \qquad \text{(1 mark)}$$

We know that $\frac{dA}{dt} = -2\pi$.

When $A = \pi r^2 = 25\pi$, we get $r = 5$. (1 mark)

Substituting $\frac{dA}{dt} = 2\pi$ and $r = 5$ into the equation (1),

we obtain $-2\pi = 2\pi \cdot 5 \cdot \frac{dr}{dt}$.

So $\frac{dr}{dt} = -\frac{1}{5} = -0.2\,(cm/s)$ (1 mark)

6. A rectangle has a perimeter of $200\,cm$. What length and width should it have so that its area is a maximum? (6 marks)

Solution Let x be the width of a rectangle, y its length and A its area.

So $A = x \cdot y$

Because the perimeter of the rectangle is $200\ cm$, we have that $2x + 2y = 200$, that is, $y = 100 - x$. (1 mark)

So $A = x \cdot (100 - x) = 100x - x^2$. (1 mark)

Since x and y should be positive, that is $x > 0$ and $100 - x > 0$.

Thus the domain of A is $(0,100)$. (1 mark)

For this question, we can take the closed interval $[0,100]$ as the domain of A.

Now, we find the maximum of A on $[0,100]$. (1 mark)

$$A' = (100x - x^2)' = 100 - 2x.$$

Let $A' = 0$, we get the critical number $x = 50$ on $(0,100)$. (1 mark)

Evaluate $A(0) = 0$,

$A(50) = 50 \cdot (100 - 50) = 2500$,

$A(100) = 0$.

So the maximum value of the area is $2500\ cm^2$ when $x = 50\ cm$ and $y = 100 - x = 100 - 50 = 50\ cm$. (1 mark)