Math 151 Section D1
Midterm 1 Fall 2006
Solutions - Version 1

1. [**3 marks**] Mark each statement **T** (True) or **F** (False):

F If $f(x) < 0$ for all x and $\lim_{x\to 0} f(x)$ exists then $\lim_{x\to 0} f(x) < 0$.

F A function f is differentiable at c if its graph has a tangent line at the point $(c, f(c))$.

F If $\lim_{x\to 1} f(x) = \infty$ and $\lim_{x\to 1} g(x) = \infty$ then $\lim_{x\to 1}[f(x)/g(x)]$ must be equal to 1.

F A function can have three different horizontal asymptotes.

T A function f is differentiable on an open interval (a, b) if it is differentiable at every number in that interval.

T If f is differentiable at c, then f is continuous at c.

2. (a) [**1**] Show by means of an example that $\lim_{x\to 1}(f(x)\cdot g(x))$ may exist even though neither $\lim_{x\to 1} f(x)$ nor $\lim_{x\to 1} g(x)$ exists. Justify your answer.

Solution: One possible example is:

$$f(x) = \begin{cases} 2 & \text{if } x \le 1 \\ \frac{1}{2} & \text{if } x > 1 \end{cases} \text{ and } g(x) = \begin{cases} \frac{1}{2} & \text{if } x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$$

Since

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} g(x) = 2$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} g(x) = 1/2$$

we see that neither $\lim_{x\to 1} f(x)$ nor $\lim_{x\to 1} g(x)$ exists. On the other hand $f(x)\cdot g(x) = 1$ for all $x \in \mathbb{R}$.Therefore $\lim_{x\to 1}(f(x)\cdot g(x)) = 1$.

(b) [**1**] Draw a graph of a function with a removable discontinuity.

Solution: See Figure 1.

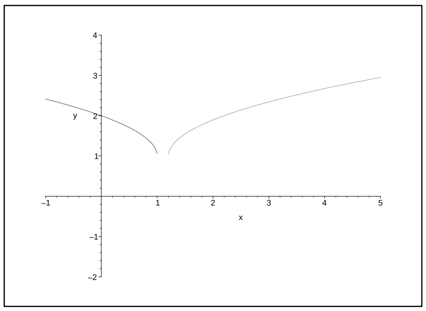

Figure 1: Removable Discontinuity

(c) [**1**] Give an example of a function f that is continuous for all real numbers and such that f' is not defined at $x = 3$. Write a formula and draw a graph of f.

Solution: One such a function is $f(x) = |x - 3|$. See Figure 2.

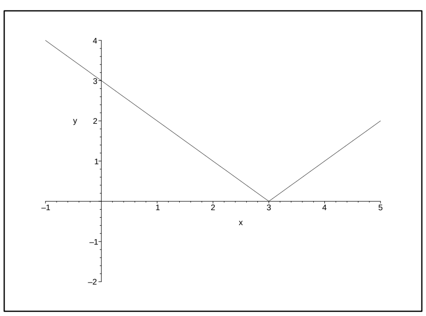

Figure 2: Continuous but Not Differentiable

3. Evaluate the following limits. **Justify your answers**.

(a) [**2**] $\lim_{x\to\pi}(x-\pi)\cos\frac{1}{\pi - x}$

Solution 1: From the fact that for all $x \neq \pi$

$$0 \leq \left|\cos \frac{1}{\pi - x}\right| \leq 1$$

it follows that for all $x \neq \pi$

$$0 \leq |x - \pi| \cdot \left|\cos \frac{1}{\pi - x}\right| = \left|(x - \pi) \cos \frac{1}{\pi - x}\right| \leq |x - \pi|.$$

Next, $\lim_{x \to \pi}(x - \pi) = 0$, by the Squeeze theorem, implies

$$\lim_{x \to \pi} \left|(x - \pi) \cos \frac{1}{\pi - x}\right| = 0.$$

Since the absolute value function is continuous we get

$$\lim_{x \to \pi} (x - \pi) \cos \frac{1}{\pi - x} = 0.$$

Solution 2: From the fact that for all $x \neq \pi$

$$-1 \leq \cos \frac{1}{\pi - x} \leq 1$$

it follows that for all $x > \pi$

$$\pi - x \leq (x - \pi) \cdot \cos \frac{1}{\pi - x} \leq x - \pi.$$

Next, $\lim_{x \to \pi}(x - \pi) = \lim_{x \to \pi}(\pi - x) = 0$, by the Squeeze theorem, implies

$$\lim_{x \to \pi^+} (x - \pi) \cos \frac{1}{\pi - x} = 0.$$

Similarly, for all $x < \pi$ from

$$\pi - x \geq (x - \pi) \cdot \cos \frac{1}{\pi - x} \geq x - \pi$$

we get

$$\lim_{x \to \pi^-} (x - \pi) \cos \frac{1}{\pi - x} = 0.$$

Therefore

$$\lim_{x \to \pi} (x - \pi) \cos \frac{1}{\pi - x} = 0.$$

(b) **[2]** $\lim_{x\to 5} \frac{25-x^2}{2x^2-17x+35}$

Solution:

$$\begin{aligned} \lim_{x\to 5} \frac{25-x^2}{2x^2-17x+35} &= \lim_{x\to 5} \frac{(5-x)(5+x)}{(x-5)(2x-7)} \\ &= \lim_{x\to 5} \frac{-(5+x)}{2x-7} = -\frac{10}{3} \end{aligned}$$

(c) **[2]** $\lim_{x\to\infty} \frac{3x^3-2x^2+2x-1}{4-2x+5x^2-2x^3}$

Solution:

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^3-2x^2+2x-1}{4-2x+5x^2-2x^3} &= \lim_{x\to\infty} \frac{3-\frac{2}{x}+\frac{2}{x^2}-\frac{1}{x^3}}{\frac{4}{x^3}-\frac{2}{x^2}+\frac{5}{x}-2} \\ &= -\frac{3}{2} \end{aligned}$$

4. (a) **[2]** State the Intermediate Value Theorem.

Solution: Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

(b) **[4]** Show that the equation $x^3 - 4x + 1 = 0$ has three different roots by calculating the values of the left-hand side at $x = -3, -2, -1, 0, 1, 2, 3$ and then applying the Intermediate Value Theorem.

Solution: Let $f(x) = x^3 - 4x + 1$. Since f is a polynomial we conclude that f is continuous on $\mathbb{R}$.

From

x	-3	-2	-1	0	1	2	3
$f(x)$	-14	1	4	1	-2	1	16

we see that $f(-3) < 0$, $f(-2) > 0$, $f(1) < 0$, and $f(2) > 0$. By the Intermediate Value Theorem, there are c_1 between -3 and -2, c_2 between -2 and 1, and c_3 between 1 and 2 such that

$$f(c_1) = f(c_2) = f(c_3) = 0.$$

Therefore c_1, c_2, and c_3 are three different roots of the given equation.

5. (a) **[2]** Define the derivative of a function f at a number a.
Solution: The derivative of a function f at a number a, denoted by $f'(a)$, is
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
if this limit exists.

(b) **[4]** Let $f(x) = \sqrt{2-x}$. Use the definition of the derivative to find $f'(-2)$.
Solution:
$$\begin{aligned} f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2-(-2+h)} - \sqrt{2-(-2)}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4-h}-2}{h} \cdot \frac{\sqrt{4-h}+2}{\sqrt{4-h}+2} \\ &= \lim_{h\to 0} \frac{4-h-4}{h(\sqrt{4-h}+2)} \\ &= \lim_{h\to 0} \frac{-1}{\sqrt{4-h}+2} = -\frac{1}{4} \end{aligned}$$

(c) **[2]** Find an equation of the tangent line to the curve $y = f(x) = \sqrt{2-x}$ at the point $(-2, f(-2))$.
Solution: From $f(-2) = \sqrt{2-(-2)} = 2$ we get that an equation of the tangent line is
$$y - 2 = -\frac{1}{4} \cdot (x+2).$$

6. Let $f(3) = 3$, $g(3) = 2$, $f'(3) = -1$ and $g'(3) = -2$. Evaluate:

(a) **[2]** $F'(3)$ if $F(x) = e^x g(x) - \sqrt[3]{x^2} f(x)$.
Solution: From
$$F'(x) = e^x g(x) + e^x g'(x) - \frac{2}{3} x^{-1/3} f(x) - \sqrt[3]{x^2} f'(x)$$

we get

$$\begin{aligned} F'(3) &= e^3(g(3)+g'(3)) - \frac{2}{3}\cdot 3^{-1/3} f(3) - \sqrt[3]{3^2} f'(3) \\ &= e^3(2-2) - \frac{2}{3\sqrt[3]{3}}\cdot 3 - \sqrt[3]{9}\cdot(-1) \\ &= -\frac{2}{\sqrt[3]{3}} + \sqrt[3]{9}. \end{aligned}$$

(b) **[2]** $G'(3)$ if $G(x) = \frac{f(x)+x^3}{1-g(x)}$.

Solution: From

$$G'(x) = \frac{(f'(x)+3x^2)(1-g(x)) - (f(x)+x^3)(-g'(x))}{(1-g(x))^2}$$

we get

$$\begin{aligned} G'(3) &= \frac{(f'(3)+3\cdot 3^2)(1-g(3)) - (f(3)+3^3)(-g'(3))}{(1-g(3))^2} \\ &= \frac{(-1+27)(1-2) - (3+27)(-(-2))}{(1-2)^2} \\ &= -86. \end{aligned}$$