MATH 151
Midterm 1 Solutions, October 5, 2005

1) (4 marks) Evaluate the following limit if it exists. If it does not exist, explain why. You must provide your reasoning which CANNOT be simply "this is what the calculator tells me."

$$\lim_{x\to 0^+} \frac{\sqrt{x}-x}{\sqrt{x}}.$$

Solution:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sqrt{x}-x}{\sqrt{x}} &= \lim_{x\to 0^+} \frac{\sqrt{x}-x}{\sqrt{x}} \cdot \frac{\sqrt{x}+x}{\sqrt{x}+x} \\
&= \lim_{x\to 0^+} \frac{x-x^2}{x+x\sqrt{x}} \\
&= \lim_{x\to 0^+} \frac{x(1-x)}{x(1+\sqrt{x})} \\
&= \lim_{x\to 0^+} \frac{1-x}{1+\sqrt{x}} \\
&= \frac{1-0}{1+\sqrt{0}} \\
&= 1
\end{aligned}$$

2a) (2 marks) Let $f(x)$ be a function. Write the definition of the derivative of $f(x)$ at $x=a$.

Solution:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

or,

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

b) (4 marks) **Using the definition in part (a)**, find $f'(2)$ for the function

$$f(x) = x^3 + 3.$$

Solution:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{[(2+h)^3 + 3] - [2^3 + 3]}{h} \\
&= \lim_{h\to 0} \frac{8 + 6h^2 + 12h + h^3 + 3 - 8 - 3}{h} \\
&= \lim_{h\to 0} \frac{h(6h + 12 + h^2)}{h} \\
&= \lim_{h\to 0} (6h + 12 + h^2) \\
&= 12
\end{aligned}$$

3) At time $t = 0$, a ball is thrown straight up in the air. The height of the ball above the ground at time t seconds is given by the function $h(t)$ (measured in meters) where

$$h(t) = 20t - 4.9t^2.$$

a) (3 marks) Using any method you like, find an expression for the velocity of the ball at time t.

Solution:

The velocity is the derivative of the position of the ball with respect to time, so it equals

$$h'(t) = 20 - 9.8t.$$

b) (2 marks) At what times is the velocity 0?

Solution:

We solve $h'(t) = 0$, i.e, $20 - 9.8t = 0$. In other words, $t = 20/9.8 = 2.040$.

c) (2 marks) What is the maximum height the ball gets above the ground? Clearly explain your answer.

Solution:

While the ball has a positive velocity, it is still gaining height. Therefore it has not reached its highest point yet. After the ball starts falling, it gets a negative velocity and it is losing height. Therefore, it has already reached its highest point.
This shows that the ball must have velocity 0 at its highest point. In (b) we have computed that this only happens at $t = 2.040$. Therefore, that must be the time when the ball reaches its highest point. The maximum height of the ball therefore is

$$h(2.040) = 20.408.$$

We conclude that the person who threw the ball is probably a Major League baseball pitcher.

4a) (3 marks) Evaluate

$$\lim_{x\to+\infty} \sin\left(\frac{1}{x^2}\right)$$

Solution:

We write $z = 1/x$. Note that as $x \to \infty$, we have $z \to 0$ from above, i.e., $z \to 0^+$. Therefore, if the limit exists, then

$$\lim_{x\to+\infty} \sin\left(\frac{1}{x^2}\right) = \lim_{z\to 0} \sin\left(z^2\right).$$

Since sin is continuous on its domain, it follows that

$$\lim_{z\to 0} \sin z^2 = \sin\left(\lim_{z\to 0} z^2\right) = \sin 0 = 0.$$

It follows that the limit indeed does exist and is 0.

b)(4 marks) Find a constant c and a function $g(x)$ such that

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 1 \\ cx + 3 & \text{if } 1 < x \leq 5 \\ g(x) & \text{if } x > 5 \end{cases}$$

is continuous at $x = 1$ AND the graph of f has a vertical asymptote at $x = 5$.

Solution:

We have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2-1) = 0$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(cx+3) = c+3$. If $f(x)$ is continuous at $x = 1$, then $\lim_{x\to 1} f(x)$ should exist and should be equal to $f(1) = 0$. This means that the one-sided limits should be equal, i.e., $0 = c + 3$. Therefore, we must have $c = -3$. Indeed, with this choice we have $\lim_{x\to 1} f(x) = f(1) = 0$.
If $g(x) = \frac{1}{x-5}$ then $\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} g(x) = \infty$. A graph of a function is said to have an asymptote at $x = a$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$. Thus, with this choice of $g(x)$, the function $f(x)$ has a vertical asymptote at $x = 5$.

5a) (1 mark) Find a positive value for δ such that the following statement is true:

$$\text{If } |x| < \delta \text{ then } x^2 < 0.04.$$

Solution:

We want x such that $x^2 < 0.04$. This will be the case if

$$-0.2 < x < 0.2.$$

Hence we could choose $\delta = 0.2$. We could of course choose any smaller number for δ (eg. $\delta = 0.1$).

b) (1 mark) Write down the precise "$\epsilon - \delta$" definition of

$$\lim_{x \to a} f(x) \; = \; L.$$

Solution:

For every $\epsilon > 0$ there exists a $\delta > 0$ such that the following statement is true:

$$\text{IF} \qquad |x - a| < \delta \qquad \text{THEN} \qquad |f(x) - L| < \epsilon.$$

c) (4 marks) Using the definition in part b), prove that

$$\lim_{x \to 1} (x^2 - 1) \; = \; 0.$$

Solution:

Step 1 (rough work): Given $\epsilon > 0$, we want to insure that $|x^2 - 1| < \epsilon$ by making $|x - 1|$ small enough. Now

$$|x^2 - 1| = |x - 1|\,|x + 1|.$$

(i) First we will choose $\delta < 1$. Hence if $|x - 1| < \delta$, we know that $x \in (0, 2)$. So $|x + 1| < 3$.
(ii) We also make sure $\delta < \epsilon/3$. Hence if $|x - 1| < \delta$, then $|x - 1| < \epsilon/3$.

Step 2 (present the proof). All the work has been done but we must summarize it coherently. Given any $\epsilon > 0$ we choose

$$\delta = \min\left(1, \frac{\epsilon}{3}\right).$$

Hence by choice of δ, if $|x - 1| < \delta$ we know that

$$|x - 1| < 1 \qquad \textbf{AND} \qquad |x - 1| < \frac{\epsilon}{3}.$$

Hence

$$|x^2 - 1| \; = \; |x - 1|\,|x + 1| \; < \; |x - 1|\,3 \; < \; \frac{\epsilon}{3}\,3 \; = \; \epsilon.$$