Simon Fraser University

Math 151-3, Summer 04
Final Exam

Date: 4 Aug. 2004
Time: 15:30 - 18:30
Place: WMC 3260

Last Name________________ Given Names______________

Student Number_________________

Instructions

1. Do not open this test booklet until instructed to do so.
2. Print your name and write your student number above.
3. No calculators or other calculating devices may be used.
4. Full marks will be awarded for correct, complete and well-organized solutions.
5. You may use the back of any page for rough work.
6. There are 13 pages in this test booklet.

Question	1	2	3	4	5
Marks	/8	/8	/10	/11	/9

Question	6	7	8	9	10
Marks	/10	/9	/11	/15	/9

Total______________/100

Good Luck!

1 a) Let $f(x)=\begin{cases}\dfrac{x^2-1}{|x-1|}, & x\neq 1\\ 4, & x=1\end{cases}$, find $\lim\limits_{x\to 1^-} f(x)$ (4 marks)

Solution $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\dfrac{x^2-1}{|x-1|}=\lim\limits_{x\to 1^-}\dfrac{x^2-1}{1-x}$ (2 marks)

$\lim\limits_{x\to 1^-}\dfrac{(x+1)(x-1)}{1-x}=\lim\limits_{x\to 1^-}(-1)(x+1)$

$=-2$ (2 marks)

b) Find $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$. (4 marks)

Solution 1 Let $y=(1+\sin x)^{\frac{1}{x}}$,

$\ln y=\ln(1+\sin x)^{\frac{1}{x}}=\dfrac{\ln(1+\sin x)}{x}$. (1 mark)

Because $\lim\limits_{x\to 0}\ln y=\lim\limits_{x\to 0}\dfrac{\ln(1+\sin x)}{x}\ (\dfrac{0}{0})=\lim\limits_{x\to 0}\dfrac{[\ln(1+\sin x)]'}{x'}$

$=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{1+\sin x}\cdot\cos x}{1}=1$, (2 marks)

$\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}=\lim\limits_{x\to 0}y=\lim\limits_{x\to 0}e^{\ln y}=e$. (1 mark)

Solution 2 $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}=\lim\limits_{x\to 0}(1+\sin x)^{\frac{\sin x}{\sin x}\cdot\frac{1}{x}}$ (2 marks)

$=\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{\sin x}\cdot\frac{\sin x}{x}}=\lim\limits_{x\to 0}\left[(1+\sin x)^{\frac{1}{\sin x}}\right]^{\frac{\sin x}{x}}$

$=e^1=e$ (2 marks)

2 a) Find the domain of the function $f(x) = \dfrac{\ln[\ln(\ln x)]}{x-3} + \sin x$. (4 marks)

Solution x should satisfy the following conditions

$$\begin{cases} x > 0 \\ \ln x > 0 \\ \ln\ln x > 0 \\ x - 3 \neq 0 \end{cases}, \quad \text{that is} \begin{cases} x > 0 \\ x > 1 \\ x > e \\ x \neq 3 \end{cases}. \qquad \text{(2 marks)}$$

Finally, the domain is the set $\{x; x > e, x \neq 3\}$. (2 marks)

b) Given $F(x) = f^2[g(x)]$, $g(1) = 2$, $g'(1) = 3$, $f(2) = 4$, and $f'(2) = 5$, find $F'(1)$. (4 marks)

Solution Using the chain rule, we have

$$F'(x) = 2f[g(x)] \cdot f'(g(x)) \cdot g'(x), \qquad \text{(2 marks)}$$

So

$$F'(1) = 2f[g(1)] \cdot f'[g(1)] \cdot g'(1) = 2f(2) \cdot f'(2) \cdot g'(1)$$
$$= 2f(2) \cdot f'(2) \cdot g'(1) = 2 \cdot 4 \cdot 5 \cdot 3 = 120. \qquad \text{(2 marks)}$$

3 a) Given $y = \tan^{-1}(\cosh\sqrt{x^2+1})$, find y'. (5 marks)

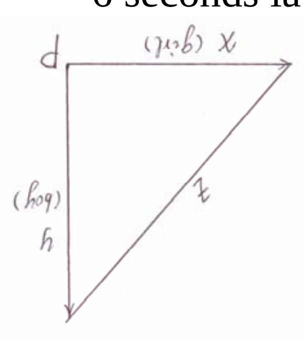
Solution $y' = \dfrac{1}{1+(\cosh\sqrt{x^2+1})^2} \cdot (\cosh\sqrt{x^2+1})'$ (2 marks)

$= \dfrac{1}{1+(\cosh\sqrt{x^2+1})^2} \cdot \sinh\sqrt{x^2+1} \cdot (\sqrt{x^2+1})'$ (1 mark)

$= \dfrac{\sinh\sqrt{x^2+1}}{1+(\cosh\sqrt{x^2+1})^2} \cdot \dfrac{(x^2)'}{2\sqrt{x^2+1}}$ (1 mark)

$= \dfrac{\sinh\sqrt{x^2+1}}{1+(\cosh\sqrt{x^2+1})^2} \cdot \dfrac{x}{\sqrt{x^2+1}}$. (1 mark)

b) A boy starts walking north at a speed of 1.5 m/s, and a girl starts walking west at the same point P at the same time at a speed of 2 m/s. At what rate is the distance between the boy and the girl increasing 6 seconds later? (5 marks)



Solution Let t be the time in seconds, $x = x(t)$ be the position function of the girl and $y = y(t)$ be the position function of the boy. The distance z between boy and girl satisfies the equation

$z^2 = x^2 + y^2$. (1 mark)

Differentiating both sides of the equation with respect to t, we have

$(z^2)'_t = (x^2+y^2)'_t$, $2z \cdot z'_t = 2x \cdot x'_t + 2y \cdot y'_t$.

So $z'_t = \dfrac{x \cdot x'_t + y \cdot y'_t}{z}$ (2 marks)

$x'_t = 2$ m/s, $y'_t = 1.5$ m/s are given.

When $t = 6\ s$, $x = 2 \cdot 6 = 12$ m, $y = 1.5 \cdot 6 = 9$ m,

$z = \sqrt{x^2+y^2} = \sqrt{12^2+9^2} = 15$ m. (1 mark)

So, $z'_t\big|_{t=6} = \left.\dfrac{x \cdot x'_t + y \cdot y'_t}{z}\right|_{t=6} = \dfrac{12 \cdot 2 + 9 \cdot 1.5}{15} = 2.5$ (m/s). (1 mark)

4 a) Use a linear approximation to estimate $\ln 0.9$. (5 marks)

Solution The linear approximation of $f(x)$ at $x=a$ is

$$f(x) \approx f(a) + f'(a)(x-a).$$

Take $f(x) = \ln x$, $a = 1$. (2 marks)

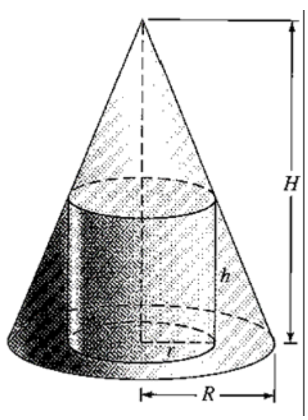
$$f'(x) = (\ln x)' = 1/x.$$

The linear approximation of $f(x) = \ln x$ at $x = 1$ is

$$\ln x \approx \ln 1 + \frac{1}{1}(x-1) = (x-1) \quad \text{(2 marks)}$$

So $\ln 0.9 \approx (0.9-1) = -0.1$. (1 mark)

b) Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius R and height H. (6 marks)



Solution Let r be the radius of the base of the right circular cylinder, h be its height.

So the volume V is $V = \pi r^2 h$. (1 mark)

By similar triangles, we get

$\dfrac{H}{R} = \dfrac{H-h}{r}$ .Solving it for h, we have

$$h = H - \frac{H}{R} r$$

Hence $V = \pi r^2 h = (\pi r^2)(H - \frac{H}{R} r) = \pi H r^2 - \pi \frac{H}{R} r^3$. (2 mark)

We take the closed interval $[0, R]$ as the domain of V.

Now, find the maximum of V on $[0, R]$.

$$V' = (\pi H r^2 - \pi \frac{H}{R} r^3)' = 2\pi H r - 3\pi \frac{H}{R} r^2$$

Let $V' = 0$, we get $r = 2R/3$ on $(0, R)$. (1 mark)

Evaluate $V(0) = 0$, $V(2R/3) = 4\pi R^2 H/27$ and $V(R) = 0$.

Thus the cylinder of maximum volume has radius $r = \dfrac{2R}{3}$, (1 mark)

and the height is $h\Big|_{r=\frac{2R}{3}} = H - \dfrac{H}{R} \cdot \dfrac{2R}{3} = \dfrac{H}{3}$. (1 mark)

5 a) Prove that $f(x)=\dfrac{1}{(x+1)^2}-2x+\sin x$ has exactly one positive root.

(5 marks)

Solution $f(x)=\dfrac{1}{(x+1)^2}-2x+\sin x$ is a continuous on $[0,1]$.

Since $f(0)=1>0$ and $f(1)=-\dfrac{7}{4}+\sin 1<0$, (2 marks)

by the intermediate value property, there exists at least one number c in $(0,1)$ such that $f(c)=0$. (1 mark)

It follows that $f(x)$ has at least one positive root.

Since $f'(x)=\dfrac{-2}{(x+1)^3}-2+\cos x<0$, for all $x>0$,

$f(x)$ is a decreasing function on $[0,+\infty)$.

Thus, $f(x)$ has exactly one positive root. (2 marks)

(Or:suppose that c_1 and c_2 are two positive roots of $f(x)$, that is

$f(c_1)=0=f(c_2)$.

By the Rolle theorem, there exist at lest one number c such that

$f'(c)=0, \quad c_1<c<c_2$.

But $f'(x)<0$, for all $x>0$, we get a contradiction,

which implies that $f(x)$ can only have one positive root)

b) Use the mean value theorem to show that

$|\sin b-\sin a|\le|b-a|$, for all real numbers a and b. (4 marks)

Solution Take $f(x)=\sin x$.

For all real numbers a and b, $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

By the mean value theorem, there exists at least a number c in (a,b) such that $f(b)-f(a)=f'(c)(b-a)$, that is

$\sin b-\sin a=\cos c(b-a)$. (2 marks)

So $|\sin b-\sin a|=|\cos c|\cdot|b-a|$.

Because $|\cos c|\le 1$, for any c, (1 mark)

$|\sin b-\sin a|\le|b-a|$. (1 mark)

6 Given the parametric curve

$$x = e^t,\ y = e^{-t}.$$

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. (6 marks)

Solution 1 $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(e^{-t})'}{(e^t)'} = \frac{-e^{-t}}{e^t} = -e^{-2t}$. (3 marks)

$$\frac{d^2y}{dx^2} = \frac{\frac{d(y_x')}{dt}}{x'(t)}$$

$$= \frac{(-e^{-2t})'}{(e^t)'} = \frac{2e^{-2t}}{e^t} = 2e^{-3t}.$$ (3 marks)

Solution 2 $y = e^{-t} = \frac{1}{e^t} = \frac{1}{x}$, (2 marks)

$\frac{dy}{dx} = -\frac{1}{x^2}$, (2 marks)

$\frac{d^2y}{dx^2} = (-\frac{1}{x^2})' = 2x^{-3}$. (2 marks)

b) Find the equation of the line tangent to the curve that is parallel to the line $y + x = 1$. (4 marks)

Solution The tangent line is parallel to the line $y + x = 1$,
so the slope of the tangent is -1. (1 mark)

Let $\frac{dy}{dx} = -1$, we have

$-e^{-2t} = -1$, so $t = 0$. (1 mark)

When $t = 0$, $x = e^0 = 1$, $y = e^{-0} = 1$, (1 mark)

the point slope equation of the tangent line is

$y - 1 = -1(x - 1)$, that is $y = -x + 2$. (1 mark)

7 a) Express the polar equation $r = \cos 2\theta$ in rectangular coordinates.(4 marks)

Solution $r = \cos 2\theta = \cos^2\theta - \sin^2\theta$.

Multiply both sides by r^2 to get

$r^3 = (r\cos\theta)^2 - (r\sin\theta)^2$. (2 marks)

Because $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$,

we have

$(x^2 + y^2)^{\frac{3}{2}} = x^2 - y^2$. (2 marks)

b) Sketch the graph of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$, and determine its foci.

(5 marks)

Solution The $x-$intercepts are $(\pm 3, 0)$.

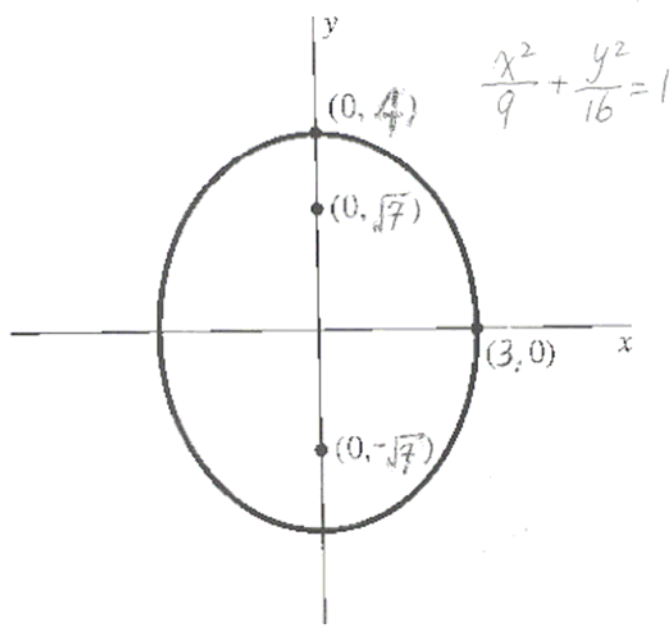
The $y-$intercepts are $(0, \pm 4)$.

So the major axis is vertical.

Take $a = 4$, $b = 3$, (1 mark)

$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$.

The foci are $(0, \pm\sqrt{7})$. (2 marks)



(2 marks)

8 a) Find the solution of the initial value problem

$$\frac{dy}{dx} = \sqrt{1-y^2}, \; y(0) = 1. \qquad \text{(6 marks)}$$

Solution Separate the variables and integrate both sides,

$$\int \frac{dy}{\sqrt{1-y^2}} = \int dx, \qquad \text{(1 mark)}$$

$$\sin^{-1} y = x + C, \qquad \text{(2 marks)}$$

Substituting $x = 0$ and $y = 1$ into the general solution,

we have $\sin^{-1}(1) = 0 + C = C$, that is $C = \pi/2$. (2 marks)

The solution is $\sin^{-1} y = x + \frac{\pi}{2}$, or

$$y = \sin(x + \frac{\pi}{2}). \qquad \text{(1 mark)}$$

b) In a certain culture of bacteria, the number of bacteria increased tenfold in 10 h. Assuming natural growth, how long did it take for their number to double? (5 marks)

Solution Let $P(t)$ be the numbers of bacteria at time t (in hours).

Assuming natural growth, the population of the bacteria is

$$P(t) = P_0 e^{kt}. \qquad \text{(2 marks)}$$

When $t = 10$ h, $P(10) = P_0 e^{k \cdot 10} = 10P_0$. So,

$$e^{10k} = 10, \; k = \frac{\ln 10}{10}. \qquad \text{(1 mark)}$$

$$P(t) = P_0 e^{\frac{\ln 10}{10}t}.$$

If $P(t) = P_0 e^{\frac{\ln 10}{10}t} = 2P_0$, then $e^{\frac{\ln 10}{10}t} = 2$. (1 mark)

So $t = \dfrac{10 \ln 2}{\ln 10}$ (h). (1 mark)

9 Given $f(x)=\dfrac{x^2-1}{x}$,

a) Find the domain and $x-$intercepts. (2 marks)

Solution The domain is $x \neq 0$. (1 mark)

Let $f(x)=\dfrac{x^2-1}{x}=0$, we get $x=\pm 1$. (1 mark)

$x-$intercepts are $(1,0)$ and $(-1,0)$.

b) Find all asymptotes. (3 marks)

Solution Because $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\dfrac{x^2-1}{x}=-\infty$,

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{x^2-1}{x}=+\infty$,

$x=0$ is a vertical asymptote. (2 marks)

Because $f(x)=\dfrac{x^2-1}{x}=x-\dfrac{1}{x}$ (or by long division)

So $y=x$ is a slant asymptote. (1 mark)

c) Determine the intervals on which the function is increasing or decreasing. Find the local maximum and minimum, if they exist. (4 marks)

Solution $f'(x)=(\dfrac{x^2-1}{x})'=(x-\dfrac{1}{x})'=1+\dfrac{1}{x^2}$. (1 mark)

The critical number is $x=0$,because $f'(0)$ does not exist.(1 mark)

Interval	$(-\infty,0)$	0	$(0,+\infty)$
k	-1		1
$f'(k)$	$+$		$+$
$f'(x)$	$+$		$+$
$f(x)$	$\nearrow$		$\nearrow$

So, f is increasing on both $(-\infty,0)$ and $(0,+\infty)$. (1 mark)

f has neither a local maximum nor a local minimum. (1 mark)

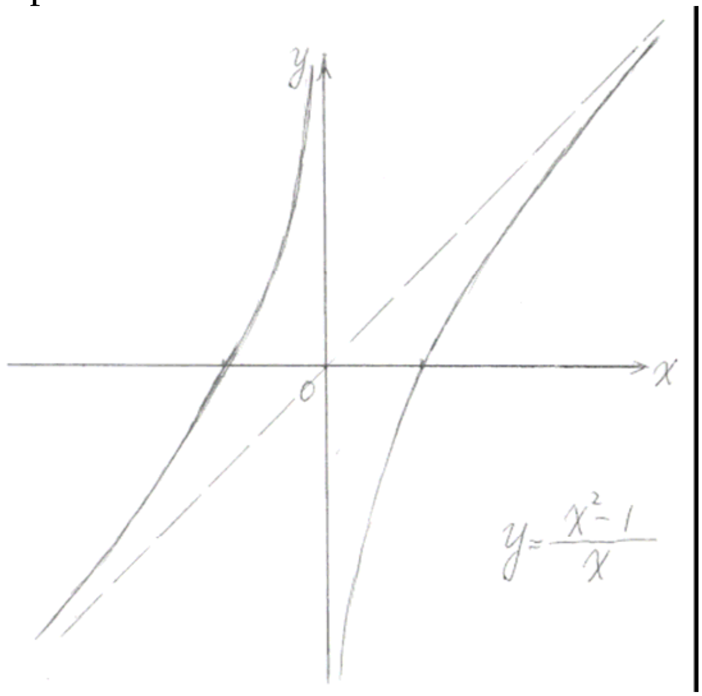
d) Determine the intervals on which the function is concave upward or downward. Find the inflection points, if they exist. (3 marks)

Solution $f''(x) = [f'(x)]' = (1 + \frac{1}{x^2})' = -\frac{2}{x^3}$

Interval	$(-\infty, 0)$	0	$(0, +\infty)$
k	-1		1
$f''(k)$	$+$		$-$
$f''(x)$	$+$		$-$
$f(x)$	$\cup$		$\cap$

So, f is concave upward on $(-\infty, 0)$,
f is concave downward on $(0, +\infty)$.
f has no inflection points.
(f'' change the sign from the left side of $x = 0$ to the right side, but f is not continuous at $x = 0$.)

e) Sketch the graph. (3 marks)



10 a) If $g(x)$ is continuous (but not differentiable) at $x=0$, $g(0)=8$, and $f(x)=x\cdot g(x)$, find $f'(0)$. (4 marks)

Solution $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{h\cdot g(h)}{h}=\lim_{h\to 0}g(h)=8.$$ (4 marks)

b) Let l be any tangent to the curve: $\sqrt{x}+\sqrt{y}=\sqrt{k}$, show that the sum of $x-$ intercept and $y-$intercept of l is k . (5 marks)

Solution Let (a,b) any fixed point on the curve $\sqrt{x}+\sqrt{y}=\sqrt{k}$.

$\sqrt{a}+\sqrt{b}=\sqrt{k}$.

In order to find the slope of the tangent line, we have to find y'.

By implicit differentiation, we have

$$(\sqrt{x}+\sqrt{y})'=(\sqrt{k})', \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0.$$

So $y'=-\frac{\sqrt{y}}{\sqrt{x}}$, $y'\big|_{(a,b)}=-\frac{\sqrt{b}}{\sqrt{a}}$.

The equation of the tangent line to the curve at (a,b) is

$$y-b=-\frac{\sqrt{b}}{\sqrt{a}}(x-a).$$ (2 marks)

Let $x=0$, we get the $y-$intercept: $y_I=b+\sqrt{ab}$,

$y=0$, we get the $x-$intercept: $x_I=a+\sqrt{ab}$. (2 marks)

Finally, we have

$$x_I+y_I=a+\sqrt{ab}+b+\sqrt{ab}=a+2\sqrt{ab}+b$$

$$=(\sqrt{a}+\sqrt{b})^2=k.$$ (1 mark)

Inverse Trigonometric Functions

Function	Domain	Range
$\sin^{-1} x$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$\tan^{-1} x$	$-\infty < x < +\infty$	$-\pi/2 < y < \pi/2$
$\cot^{-1} x$	$-\infty < x < +\infty$	$0 < y < \pi$
$\sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y < \pi/2,$ $\pi/2 < y \le \pi$
$\csc^{-1} x$	$\lvert x \rvert \ge 1$	$-\pi/2 \le y < 0,$ $0 < y \le \pi/2$

Some Derivative Formulas

$$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}, \ |x|>1, \qquad (\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}, \ |x|>1,$$

$$(\sinh x)' = \cosh x, \qquad (\cosh x)' = \sinh x,$$

$$(\tanh x)' = \operatorname{sech}^2 x, \qquad (\coth x)' = -\operatorname{csch}^2 x,$$

$$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x, \qquad (\operatorname{csch} x)' = -\operatorname{csch} x \coth x.$$

Hyperbolic Functions Identities

$$\cosh^2 x - \sinh^2 x = 1,$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x, \qquad \coth^2 x - 1 = \operatorname{csch}^2 x,$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh 2x = 2 \sinh x \cosh x, \qquad \cosh 2x = \cosh^2 x + \sinh^2 x,$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1), \qquad \sinh^2 x = \frac{1}{2}(\cosh 2x - 1).$$