Simon Fraser University
Department of Mathematics
Burnaby and Surrey Campus

MATH 151-3, Spring 2005
Final
April 16th, 2005, 3:30 – 6:30 pm

Last Name (please print): _____

First Name (please print): _____

Student Number: _____

Instructions:

1. DO NOT OPEN THIS BOOKLET UNTIL TOLD TO DO SO.
2. Fill in the above box.
3. This exam contains … pages with a total of 9 questions. Once the exam begins please check to make sure your exam is complete.
4. SHOW ALL YOUR WORK!
5. If you run out of space in a problem, use the space on the back of the previous page and clearly indicate where the solution continues.
6. **Only** scientific calculators are allowed (basic math/stat functions + memory key).
7. No book, paper, or device, other than the usual writing instruments, this booklet and a scientific calculator, shall be within reach of a student during the examination.
8. During the examination, speaking to, communicating with, or deliberately exposing written papers to the view of other examinees is forbidden.
9. Try your best!

Do not write in this table!	
Question	**Marks**
1	**/10**
2	**/18**
3	**/17**
4	**/8**
5	**/15**
6	**/8**
7	**/8**
8	**/8**
9	**/8**
Total	**/100**

1.

(a) [2 marks] What is meant by saying that L is the *limit of* $f(x)$ *as* x *approaches* a ?

[$f(x)$ is defined for all x in some neighbourhood of a except possibly at a itself, and $f(x)$ can be made as close to L as desired by choosing x sufficiently close to a.]

(b) [2 marks] What is meant by saying that the function $f(x)$ is *continuous* at $x = a$?

[$\lim_{x \to a} f(x)$ and $f(a)$ are both defined, and $\lim_{x \to a} f(x) = f(a)$]

(c) [2 marks] State two properties that a continuous function $f(x)$ can have, either of which guarantees the function is not differentiable at $x = a$. Draw an example of each.

[Any two of: a corner at $x = a$, a vertical tangent line at $x = a$, and a cusp at $x = a$.]

(d) [2 marks] State Newton's iterative formula that gives a sequence of approximations $x_0, x_1, x_2, \ldots$ to a solution of $f(x) = 0$, assuming that x_0 is given.

[$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$]

(e) [2 marks] Draw a labelled diagram showing an example of a function $f(x)$ for which Newton's iterative formula fails to find a solution of $f(x) = 0$. Mark on your diagram x_0, x_1 and x_2.

2. Evaluate the following limits (if they exist). Show all your working.

(a) [3 marks] $\lim_{x\to 1} f(x)$, where $2x-1 \le f(x) \le x^2$ for all x in the interval (0, 2).

[1, using Squeeze Law.]

(b) [3 marks] $\lim_{x\to -\infty} \dfrac{5x^7 - 3x^5 + 1}{2x^7 + 6x^6 - 3}$

[limit of rational functions]

(c) [3 marks] $\lim_{x\to 8} \dfrac{(x-8)(x+2)}{|x-8|}$

[does not exist]

(d) [3 marks] $\lim_{x\to 0^+} (\sin x)(\ln \sin x)$

[$0.\infty$

C04S09.004: $\lim_{x\to 0+} (\sin x)(\ln \sin x) = \lim_{x\to 0+} \dfrac{\ln \sin x}{\csc x} = \lim_{x\to 0+} \dfrac{\cos x}{-\csc x \cot x \sin x} = \lim_{x\to 0+} (-\sin x) = 0.$

]

(e) [3 marks] $\lim_{x\to\infty} (x + \sin x)^{1/x}$

[∞^0. Should state "using continuity" in solution when switching ln and lim.

$$\ln\left(\lim_{x\to\infty} (x+\sin x)^{1/x}\right) = \lim_{x\to\infty} \frac{\ln(x+\sin x)}{x} = \lim_{x\to\infty} \frac{1+\cos x}{x+\sin x} = 0,$$

and therefore $\lim_{x\to\infty} (x+\sin x)^{1/x} = e^0 = 1.$

]

(f) [3 marks] $\lim_{x\to 0^+} (x + \sin x)^{1/x}$

[0^∞ : not an indeterminate form. As $x \to 0^+$, $x + \sin x \to 0$ through positive values and $1/x \to \infty$.]

3. The following questions involve derivatives.

(a) [2 marks] Evaluate $D_t \cos^{-1}(\cosh(e^{-3t}))$, without simplifying your answer.

(b) [5 marks] Use logarithmic differentiation to find $y'(u)$ as a function of u alone, where $y(u) = \left(\dfrac{(u+1)(u+2)}{(u^2+1)(u^2+2)} \right)^{1/3}$, without simplifying your answer.

[

C03S0M.054: Given: $y = \left[\dfrac{(x+1)(x+2)}{(x^2+1)(x^2+2)} \right]^{1/3}$. Then

$$\ln y = \frac{1}{3}\left[\ln(x+1) + \ln(x+2) - \ln(x^2+1) - \ln(x^2+2)\right];$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{x^2+1} - \frac{2x}{x^2+2}\right);$$

$$\frac{dy}{dx} = y(x) \cdot \frac{6 - 8x - 9x^2 - 8x^3 - 9x^4 - 2x^5}{3(x+1)(x+2)(x^2+1)(x^2+2)};$$

]

(c) [5 marks] Solve the initial value problem $\dfrac{dx}{dt} = \dfrac{36}{(4t-7)^4}$, $x(2) = 1$.

[$x(t) = \dfrac{-3}{(4t-7)^3} + 4$]

(d) [5 marks] Let $x = 2\sin t + 1$ and $y = 2t^3 - 3$ define a parametric curve. Find $\dfrac{d^2y}{dx^2}$ as a function of *t*, without simplifying your answer.

[Not too many marks for finding $\dfrac{d^2y}{dx^2}$ from $\dfrac{dy}{dx}$]

4. The equation $e^y + y(x-2) = x^2 - 8$ defines *y* implicitly as a function of *x* near a point (3, 0).

(a) [4 marks] Determine the value of y' at this point.

(b) [4 marks] Use a linear approximation to estimate the value of *y* when *x* = 2.98.

5. Let $f(x) = \dfrac{2x}{x^2 - 1}$ be defined for all $x \neq \pm 1$. You can make use of the following facts:

- $f'(x) = \dfrac{-2(x^2 + 1)}{(x^2 - 1)^2}$
- $f''(x) = \dfrac{4x(3 + x^2)}{(x^2 - 1)^2}$.

Showing all your work, determine for the graph of $y = f(x)$:

(a) [1 mark] The (x, y) co-ordinates of all intercepts.

(b) [4 marks] All asymptotes. For each vertical asymptote, if any, determine the behaviour of $f(x)$ as x approaches the vertical asymptote from the left and from the right.

(c) [1 mark] The (x, y) co-ordinates of all critical points, if any.

(d) [2 marks] The intervals on which f is increasing and the intervals on which f is decreasing.

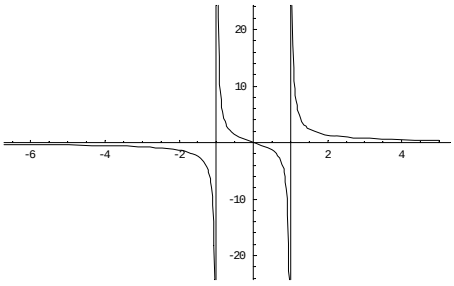
(e) [1 mark] The classification of each critical point, if any, as a minimum or maximum, local or global, or not an extremum.

(f) [2 marks] The intervals on which f is concave up and the intervals on which f is concave down.

(g) [1 mark] The (x, y) co-ordinates of all inflection points, if any.

(h) [3 marks] Sketch the graph using all the above information and label all relevant points and lines.

[



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6. [8 marks] Each rectangular page of a book must contain 30 cm^2 of printed text, and each page must have 2 cm margins at top and bottom, and a 1 cm margin at each side. What is the minimum possible area of such a page?

[

C04S04.040: If the print width is x and its height is y (in inches), then the page area is $A = (x+2)(y+4)$. We are to minimize A given $xy = 30$. Because $y = 30/x$,

$$A = A(x) = 4x + 38 + \frac{60}{x}, \quad x > 0.$$

Now

$$A'(x) = 4 - \frac{60}{x^2};$$

$A'(x) = 0$ when $x = \sqrt{15}$. But $A'(x) > 0$ for $x > \sqrt{15}$ whereas $A'(x) < 0$ for $0 < x < \sqrt{15}$. Therefore $x = \sqrt{15}$ yields the global minimum value of $A(x)$, which is $38 + 8\sqrt{15}$, approximately 68.98 square inches.

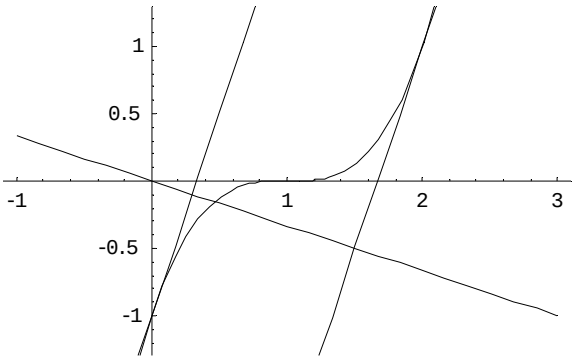
]

7. Let C be the curve $y = (x - 1)^3$ and let L be the line $3y + x = 0$.

   (a) [5 marks] Find the equation of all lines that are tangent to C and are simultaneously perpendicular to L.

   (b) [3 marks] Draw a labelled diagram showing the curve C, the line L, and the line(s) of your solution to part (a). For each line of your solution, mark on the diagram the point where it is tangent to C and (without necessarily calculating the co-ordinates) the point where it is perpendicular to L.

[Points of tangency (0, –1) and (2, 1), lines $y + 1 = 3x$ and $y - 1 = 3(x - 2)$.
Diagram should mark right angle where solution lines intersect L.



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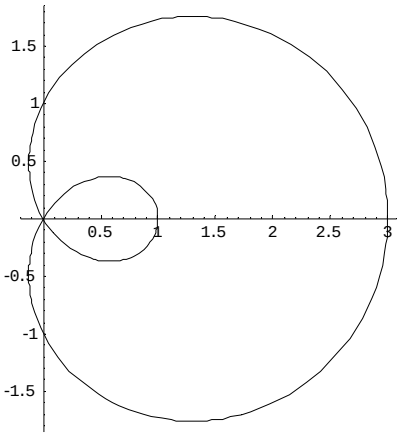
8. [8 marks] A helicopter takes off from a point 80 m away from an observer located on the ground, and rises vertically at 2 m/s. At what rate is the elevation angle of the observer's line of sight to the helicopter changing when the helicopter is 60 m above the ground?

[2/125 rad/s: use related rates or take arctan and differentiate directly.]

9. [8 marks] Sketch the curve whose polar equation is $r = -1 + 2\cos\theta$, indicating any symmetries. Mark on your sketch the polar co-ordinates of all points where the curve intersects the polar axis.

[



Symmetric about polar axis (*x*-axis).
Intersects polar axis at
(–1, 0) (equivalently (1, π))
(0, $\pm\,\pi/3$): OK to have only one of these since it's the same point
(–3, π) (equivalently (3, 0))
]