Simon Fraser University
Math 151-3, Spring 2004
Final Examination Solution (Blue version)

1. Evaluate the following limits if they exist.

   (a) [4 marks] $\lim_{x\to-\infty} \frac{2x+3x^3}{x^3+2x-1}$
   Answer

$$\lim_{x\to-\infty} \frac{2x+3x^3}{x^3+2x-1} = \lim_{x\to-\infty} \frac{\frac{2}{x^2}+3}{1+\frac{2}{x^2}-\frac{1}{x^3}} = 3.$$

   (b) [4 marks] $\lim_{x\to0^+} (1+7x)^{1/(5x)}$
   Answer

$$\begin{aligned}\lim_{x\to0^+} (1+7x)^{1/(5x)} &= \exp\left[\lim_{x\to0^+} \frac{\ln(1+7x)}{5x}\right] \\ &= \exp\left[\lim_{x\to0^+} \frac{\frac{7}{1+7x}}{5}\right] \qquad \text{by L'Hopital's Rule, since } \lim_{x\to0^+} \ln(1+7x) = \lim_{x\to0^+} (5x) = 0 \\ &= e^{7/5}.\end{aligned}$$

2. Suppose the functions $F(x)$ and $G(x)$ satisfy the following properties:

$$\begin{array}{lll} F(3)=2, & G(3)=4, & G(0)=3, \\ F'(3)=-1, & G'(3)=0, & G'(0)=2. \end{array}$$

   (a) [2 marks] If $S(x) = \frac{F(x)}{G(x)}$, find $S'(3)$. Simplify your answer.
   Answer

$$S'(3) = \frac{F'(3)\,G(3) - F(3)\,G'(3)}{G^2(3)} = \frac{-1\cdot4-2\cdot0}{4^2} = -\frac{1}{4}.$$

   (b) [2 marks] If $T(x) = F[G(x)]$, find $T'(0)$. Simplify your answer.
   Answer

$$T'(0) = F'[G(0)]\,G'(0) = F'(3)\cdot2 = -2.$$

   (c) [2 marks] If $U(x) = \ln[F(x)]$, find $U'(3)$. Simplify your answer.
   Answer

$$U'(3) = \frac{F'(3)}{F(3)} = -\frac{1}{2}.$$

3. (a) [4 marks] Find $\frac{dy}{dx}$ if $y = \sec[\operatorname{sech}(x)]$.
   Answer

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\sec[\operatorname{sech}(x)]}{d(\operatorname{sech}(x))}\cdot\frac{d(\operatorname{sech}(x))}{dx} \\ &= \sec[\operatorname{sech}(x)]\tan[\operatorname{sech}(x)]\cdot(-\operatorname{sech}(x)\tanh(x))\end{aligned}$$

   (b) [4 marks] Find $\frac{dy}{dx}$ if $\ln(x+y) = xy - y^3$. Express $\frac{dy}{dx}$ as a function of x and y.
   Answer

$$\begin{aligned}\frac{d}{dx}\ln(x+y) = \frac{d}{dx}\left(xy-y^3\right) &\implies \frac{1}{x+y}\left(1+\frac{dy}{dx}\right) = y + x\frac{dy}{dx} - 3y^2\frac{dy}{dx} \\ &\implies \frac{dy}{dx} = \frac{\frac{1}{x+y}-y}{x-3y^2-\frac{1}{x+y}}\end{aligned}$$

4. (a) [4 marks] Find the equation of the line that is tangent to the graph of $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}$ at $x = 1$.

Answer Since $y' = \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{x^3}}$, the equation of the tangent is:

$$\frac{y - y(1)}{x-1} = y'(1) \implies \frac{y}{x-1} = 1 \implies y = x - 1.$$

(b) [4 marks] Use linear approximation to estimate the value of $\sqrt[3]{26^2}$. Express your answer as a single fraction (for example, 16/729).

Answer Let $f(x) = \sqrt[3]{x^2} = x^{2/3}$, whence $f'(x) = \dfrac{2}{3x^{1/3}}$.

$$\begin{aligned} & f(x) \approx f(27) + f'(x)(x-27) \quad \text{if } x \approx 27 \\ \implies & \sqrt[3]{x^2} \approx 9 + \frac{2}{9}(x-27) \quad \text{if } x \approx 27 \\ \implies & \sqrt[3]{26^2} \approx 9 + \frac{2}{9}(-1) = \frac{79}{9}. \end{aligned}$$

5. An open-top box is to have a square base and a volume of 10 m^3. The cost per square meter of material is \$5 for the bottom and \$2 for the four sides. Let x and y be the lengths of the box's width and height respectively. Let C be the total cost of material required to make the box.

(a) [4 marks] Express C as a function of x and find its domain.

Answer Given the volume of the box is $10 = x^2 y$ so that $y = \dfrac{10}{x^2}$, the cost is:

$$C = 5x^2 + 2(4xy) = 5x^2 + 8x\left(\frac{10}{x^2}\right) = 5x^2 + \frac{80}{x}.$$

The domain of C is all $x > 0$.

(b) [6 marks] Find the dimensions of the box so that the cost of materials is minimized. What is this minimum cost? Justify your answer.

Answer Since $C'(x) = 10x - \dfrac{80}{x^2}$ is continuous on $x > 0$, and since $C'(x) = 0$ only when $x = 2$, the critical value of $C(x)$ is $x = 2$. First Derivative Test:

$C'(x)$	dne	$-$	0	$+$
$C(x)$	dne	$\searrow$	60	$\nearrow$
x	≤ 0		2	

Therefore $C(x)$ has an absolute minimum when $x = 2$. Therefore the dimensions that require the minimum cost are $x = 2$ and $y = \dfrac{10}{2^2} = \dfrac{5}{2}$. The minimum cost is $C(2) = 60$ dollars.

6. For this question, consider the function

$$f(x) = \frac{x}{x^2 - 1}.$$

For your convenience, the first and second derivatives of $f(x)$ are

$$f'(x) = \frac{-x^2 - 1}{(x^2-1)^2} \quad \text{and} \quad f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

respectively.

(a) [4 marks] Determine any horizontal and vertical asymptotes of f.

Answer Since $\lim_{x \to \pm\infty} f(x) = 0$, the function $f(x)$ has one horizontal asymptote $y = 0$. Since

$$\lim_{x \to 1^+} f(x) = \lim_{x \to -1^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to 1^-} f(x) = \lim_{x \to -1^-} f(x) = -\infty,$$

the function $f(x)$ has vertical asymptotes $x = \pm 1$.

(b) [4 marks] Determine the open intervals on which f is increasing as well as those on which f is decreasing.

Answer Since $f'(x) = \dfrac{-(x^2+1)}{(x^2-1)^2}$, it is trivial that $f'(x) < 0$ for all $x \neq \pm 1$:

$f'(x)$	$-$	dne	$-$	dne	$-$
$f(x)$	$\searrow$	dne	$\searrow$	dne	$\searrow$
x		-1		1	

Therefore $f(x)$ is decreasing on $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$.

(c) [4 marks] Determine the open intervals on which f is concave upward as well as those on which f is concave downward.

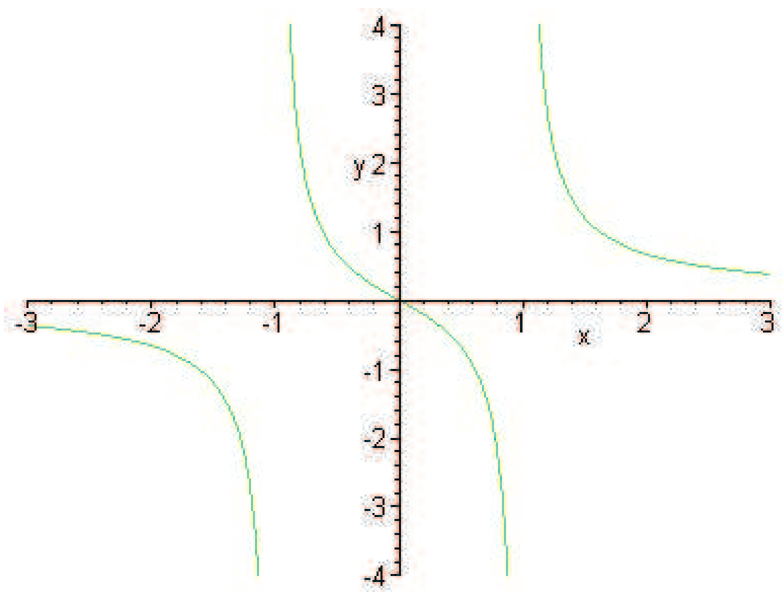
Answer Clearly $f''(x)$ is 0 only at $x = 0$, and $f''(x)$ is not defined at $x = \pm 1$.

$f''(x)$	$-$	dne	$+$	0	$-$	dne	$+$
$f(x)$	$\cap$	dne	$\cup$	0	$\cap$	dne	$\cup$
x		-1		0		1	

Therefore $f(x)$ is concave upward on $(-1,0)$ and $(1,\infty)$, and it is concave downward on $(-\infty,-1)$ and $(0,1)$.

(d) [4 marks] Based on the information found in Parts (a) to (c), sketch the graph of f. Indicate any relative extremum, inflection points and intercepts on your graph.

Answer



The origin $(0,0)$ is both an inflection point and intercept of the graph of $f(x)$.

7. (a) [4 marks] Find the function $y(x)$ that is the solution of the following initial value problem:

$$\frac{dy}{dx} = y^2 + 1 \quad \text{and} \quad y(\pi/4) = 0.$$

Answer

$$\frac{dy}{dx} = y^2 + 1 \quad \Longrightarrow \quad \int \frac{dy}{y^2+1} = \int dx \quad \Longrightarrow \quad \arctan y = x + C.$$

Since

$$y\left(\frac{\pi}{4}\right) = 0 \quad \Longrightarrow \quad \arctan 0 = \frac{\pi}{4} + C \quad \Longrightarrow \quad C = -\frac{\pi}{4},$$

we have $y = \tan\left(x - \dfrac{\pi}{4}\right)$.

(b) [4 marks] On a certain day, a scientist had 1 kg of a radioactive substance X at 1:00 pm. After six hours, only 27 g of the substance remained. How much substance X was there at 3:00 pm that same day?

Answer Suppose $t = 0$ at 1:00 pm. Given $N_0 = 1000$ g, the amount of substance X at time t is $N(t) = 1000e^{-kt}$.

$$27 = N(6) = 1000e^{-6k} \quad \Longrightarrow \quad e^{-k} = \left(\frac{27}{1000}\right)^{1/6} = \left(\frac{3}{10}\right)^{1/2} \quad \Longrightarrow \quad N(t) = 1000\left(\frac{3}{10}\right)^{t/2}.$$

Therefore $N(2) = 300$, whence there is 300 g of substance X at 3:00 pm.

8. Given the parametric curve

$$x = \cos^3 t, \qquad y = \sin^3 t.$$

(a) [4 marks] **Without** eliminating the parameter t, show that $\dfrac{dy}{dx} = -\tan t$.

Answer

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\sin^2 t \cdot \cos t}{3\cos^2 t \cdot -\sin t} = -\tan t.$$

(b) [4 marks] Determine the concavity of this curve when $t = 1$.

Answer Since

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\sec^2 t}{3\cos^2 t \cdot -\sin t} = \frac{1}{3\cos^4 t \sin t} > 0 \quad \text{when } t = 1,$$

the curve is concave upward when $t = 1$.

9. For this question, let $\mathcal{C}$ be the conic section described by the equation

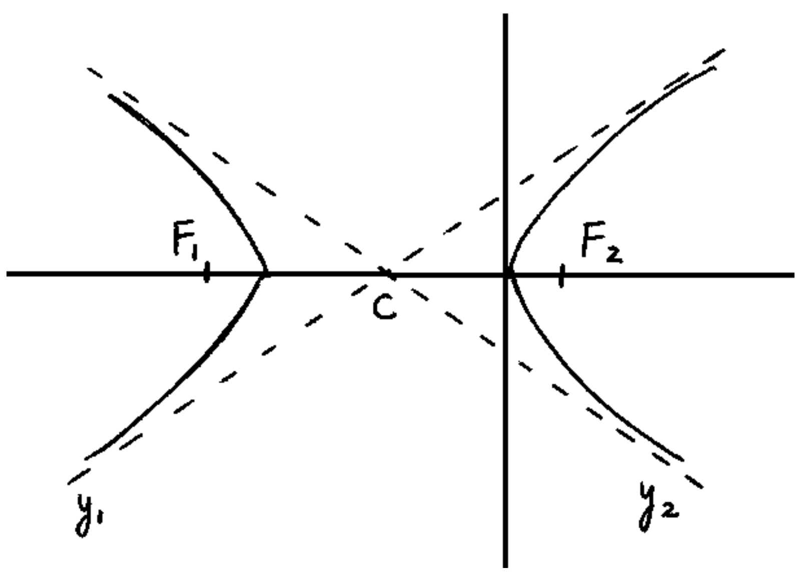
$$x^2 - 2y^2 + 4x = 0.$$

(a) [4 marks] Using the method of completing the square, identify the conic section $\mathcal{C}$.

Answer $\dfrac{(x+2)^2}{4} - \dfrac{y^2}{2} = 1$ (Hyperbola)

(b) [6 marks] Sketch the graph of $\mathcal{C}$. Indicate its center, foci and asymptotes (if any).

Answer



$$\begin{aligned}
\text{Center}: &\quad (-2, 0)\\
\text{Foci}: &\quad F_1 = \left(-\sqrt{6} - 2, 0\right), \quad F_2 = \left(\sqrt{6} - 2\right)\\
\text{Asymptotes}: &\quad y_1 = \frac{1}{\sqrt{2}}x + \sqrt{2}, \quad y_2 = -\frac{1}{\sqrt{2}}x - \sqrt{2}
\end{aligned}$$

10. (a) [4 marks] Use the **definition of limits** to prove that

$$\lim_{x \to 0} x^3 = 0.$$

Your proof should involve the variables δ and ϵ.

Answer Let $\epsilon > 0$. Note that

$$\left|x^3 - 0\right| < \epsilon \iff |x|^3 < \epsilon \iff |x - 0| < \sqrt[3]{\epsilon}.$$

Therefore letting $\delta = \sqrt[3]{\epsilon}$, we have $0 < |x - 0| < \sqrt[3]{\epsilon} \implies \left|x^3 - 0\right| < \epsilon$.

(b) [4 marks] Let f be a function that is continuous everywhere and let

$$F(x) = \begin{cases} \dfrac{f(x)\sin^2 x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Use the **definition of derivatives** to evaluate $F'(0)$. Your answer should be in terms of f. Justify your answer.

Answer Note that $\lim\limits_{h \to 0} f(h) = f(0)$ since f is continuous everywhere. Therefore

$$\begin{aligned}
F'(0) &= \lim_{h \to 0} \frac{F(0 + h) - F(0)}{h}\\
&= \lim_{h \to 0} \frac{\frac{f(h)\sin^2 h}{h} - 0}{h}\\
&= \lim_{h \to 0} \frac{f(h)\sin^2 h}{h^2}\\
&= \lim_{h \to 0} f(h) \cdot \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{\sin h}{h}\\
&= f(0) \cdot 1 \cdot 1\\
&= f(0).
\end{aligned}$$