1. **[8 marks] Define the following terms:**

   (a) The limit of $f(x)$, as x approaches a

   Solution: We write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

   (b) The derivative of a function f at a number a

   Solution: The derivative of a function f at a number a, denoted by $f'(a)$, is

   $$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

   if this limit exists.

   (c) Critical number of a function f

   Solution: A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ dos not exists.

   (d) Antiderivative of a function f

   Solution: A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

2. **[12 marks] State the following theorems:**

   (a) The Intermediate Value Theorem

   Solution: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f() \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.

   (b) The Extreme Value Theorem

   Solution: If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

   (c) Fermat's Theorem

   Solution: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

(d) The Mean Value Theorem

Solution: Let f be a function that satisfies the following hypotheses:

i. f is continuous on the closed interval $[a, b]$.

ii. f is differentiable on the open interval (a, b).

. Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$.

3. [**12 marks**] **Give an example for the each of the following:**

(a) Function $F = f \cdot g$ so that the limits of F and f at a exist and the limit of g at a does not exist.

Solution: $f(x) = x$, $g(x) = \sin(1/x)$, $F(x) = F(x) \cdot g(x) = x \sin(1/x)$

(b) Function that is continuous but not differentiable at a point.

Solution: $f(x) = |x|$

(c) Function with a critical number but no local maximum or minimum.

Solution: $f(x) = x^3$

(d) Function with a local minimum at which its second derivative equals 0.

Solution: $f(x) = x^4$

4. [**12 marks**] **Evaluate the following limits:**

(a) $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right)$

Solution: From the fact that, for any $x \neq 0$, $|\sin(1/x^2)| \leq 1$ it follows that

$$0 \leq \left| x \sin\left(\frac{1}{x^2}\right) \right| \leq |x|$$

for all $x \neq 0$. Since $\lim_{x\to 0} x = 0$ and since $f(x) = |x|$ is a continuous function, by the Squeeze Theorem it follows that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0.$$

(b) $\lim_{x\to 0} \frac{x^3 \sin\left(\frac{1}{x^2}\right)}{\sin x}$

Solution: We note that

$$\frac{x^3 \sin\left(\frac{1}{x^2}\right)}{\sin x} = \frac{x}{\sin x} \cdot x \cdot x \sin\left(\frac{1}{x^2}\right).$$

From the facts that $\lim_{x\to 0} \frac{x}{\sin x} = 1$, $\lim_{x\to 0} x = 0$, and, by the previous part of this question, $\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0$ it follows that

$$\lim_{x\to 0} \frac{x^3 \sin\left(\frac{1}{x^2}\right)}{\sin x} = 0.$$

(c) $\lim_{x\to 0^+} x^{\tan x}$

Solution: We note that the given limit is an indeterminate form 0^0. From the fact that, for $x > 0$, $x^{\tan x} = e^{\ln\left(x^{\tan x}\right)} = e^{\tan x \ln x}$ and the fact that the function $f(x) = e^x$ is continuous for all x, it follows that

$$\lim_{x\to 0^+} x^{\tan x} = \lim_{x\to 0^+} e^{\tan x \ln x} = e^{\lim_{x\to 0^+} \tan x \ln x}.$$

Next we note that

$$\lim_{x\to 0^+} \tan x \ln x = \lim_{x\to 0+} \frac{\sin x \ln x}{\cos x}$$

and since $\lim_{x\to 0} \cos x = 1$ we conclude that it is enough to find $\lim_{x\to 0^+} \sin x \ln x$.

Since $\lim_{x\to 0^+} \sin x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$ to use l'Hospital's rule we write

$$\sin x \ln x = \frac{\ln x}{\frac{1}{\sin x}}.$$

Thus, by l'Hospital's rule

$$\begin{aligned}
\lim_{x\to 0^+} \sin x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} \\
&= \lim_{x\to 0^+} \left(\sin x \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos x}\right) = 0 \cdot (-1) \cdot 1 = 0.
\end{aligned}$$

Finally, $\lim_{x\to 0^+} x^{\tan x} = e^0 = 1$.

5. [**12 marks**] **Find the derivative $\mathbf{y' = \frac{dy}{dx}}$:**

(a) $y = x^3 + 3^x + x^{3x}$

Solution: We note that $\frac{d}{dx}(x^3) = 3x^2$ and that $\frac{d}{dx}(3^x) = 3^x \ln 3$. To find the derivative of the function $z = x^{3x}$ we use logarithmic differentiation:

$$\begin{aligned} z &= x^{3x} \\ \ln z &= 3x \ln x \\ \frac{z'}{z} &= 3 \ln x + 3x \cdot \frac{1}{x} = 3 \ln ex \\ z' &= 3z \ln ex = 3x^{3x} \ln ex \end{aligned}$$

Thus $\frac{d}{dx}\left(x^3 + 3^x + x^{3x}\right) = 3x^2 + 3^x \ln 3 + 3x^{3x} \ln ex$.

(b) $y = e^{-5x} \cosh 3x$

Solution: From

$$y = e^{-5x} \cosh 3x = e^{-5x} \cdot \frac{e^{3x} + e^{-3x}}{2} = \frac{1}{2} \cdot (e^{-2x} + e^{-8x})$$

we get that

$$y' = \frac{1}{2} \cdot (-2e^{-2x} - 8e^{-8x}) = -e^{-2x} - 4e^{-8x}.$$

(c) $\tan^{-1} \frac{y}{x} = \frac{1}{2} \ln(x^2 + y^2)$

Solution: We use implicit differentiation:

$$\begin{aligned} \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{xy' - y}{x^2} &= \frac{1}{2} \cdot \frac{2x + 2yy'}{x^2 + y^2} \\ \frac{xy' - y}{x^2 + y^2} &= \frac{x + yy'}{x^2 + y^2} \\ xy' - y &= x + yy' \\ (x - y)y' &= x + y \\ y' &= \tfrac{x+y}{y-x} \end{aligned}$$

(d) $y = \frac{x^5 e^{x^3} \sqrt[3]{x^2+1}}{(x+1)^4}$

Solution: We take logarithms of both sides:

$$
\begin{aligned}
\ln y &= 5 \ln x + x^3 + \frac{1}{3} \cdot \ln(x^2+1) - 4\ln(x+1) \\
\frac{y'}{y} &= \frac{5}{x} + 3x^2 + \frac{1}{3} \cdot \frac{2x}{x^2+1} - \frac{4}{x+1} \\
y' &= y \cdot \left(\frac{5}{x} + 3x^2 + \frac{2x}{3(x^2+1)} - \frac{4}{x+1} \right) \\
y' &= \frac{x^5 e^{x^3} \sqrt[3]{x^2+1}}{(x+1)^4} \cdot \left(\frac{5}{x} + 3x^2 + \frac{2x}{3(x^2+1)} - \frac{4}{x+1} \right)
\end{aligned}
$$

6. [**5 marks**] A water tank is in the shape of a cone with vertical axis and vertex downward. The tank has radius 3 m and is 5 m high. At first the tank is full of water, but at time $t = 0$ (in seconds), a small hole at the vertex is opened and the water begins to drain. When the height of water in the tank has dropped to 3 m, the water is flowing out at 2 m^3/s. At what rate, in meters per second, is the water level dropping then?

   [**Note:** The volume of a cone with the radius r and the hight H is given by $V = \frac{\pi r^2 H}{3}$.]

   Solution: Let O be the vertex of the cone an let A be the center of the circle that is the boundary of the top of the tank. Let B be a point on the circle. It is given that the length of the line segment OA equals $|OA| = 5$ m and that the length of the line segment AB equals $|AB| = 3$ m.

   Next, let x be the height of water at time t. The question is to find $\frac{dx}{dt}$ when $x = 3$ m if it is known that at that moment $\frac{dV}{dt} = -2$ m^3/s, where V represents the volume of water at time t.

   Let C be the point on the line segment OA such that $|OC| = x$ and let D be the point on the cone such that the line segment CD is perpendicular to the line segment OC and parallel to the line segment AB. Clearly, the point D belongs to the circle that is the boundary of the top of the cone formed by water in the tank when the hight of water is x meters. The radius of that circle is $|CD|$, call this number r.

We note that OAB and OCD are two similar triangles. This implies

$$\frac{|OC|}{|OA|} = \frac{|CD|}{|AB|}$$

what is the same as

$$\frac{x}{5} = \frac{r}{3}.$$

Thus $r = \frac{3x}{5}$.

It follows that the volume of water at the moment when water is x meters high equals

$$V = \frac{r^2 \pi x}{3} = \frac{3x^3\pi}{25}.$$

Thus

$$\frac{dV}{dt} = \frac{9x^2\pi}{25} \cdot \frac{dx}{dt}$$

which implies that, when $x = 3$ m

$$-2 = \frac{9 \cdot 9\pi}{25} \cdot \frac{dx}{dt}.$$

Finally,

$$\frac{dx}{dt} = -\frac{50}{81\pi} \text{ m/s}.$$

7. [**5 marks**] Use the linear approximation to approximate $(63)^{2/3}$. Then use differentials to estimate the error.

Solution: Let $f(x) = x^{2/3}$. We note that $f(64) = 16$. Also we note that $f'(x) = \frac{2}{3} \cdot x^{-1/3}$ and $f'(64) = \frac{2}{3} \cdot 64^{-1/3} = \frac{1}{6}$. The linearization of f at $a = 64$ is given by

$$L(x) = 16 + \frac{1}{6} \cdot (x - 64).$$

Thus $L(x) = \frac{x}{6} + \frac{16}{3}$. It follows that

$$(63)^{2/3} = f(63) \approx L(63) = \frac{63}{6} + \frac{16}{3} = \frac{95}{6}.$$

To estimate the error we use differentials:

$$|f(63) - L(63)| \approx |dy| = |f'(64) \cdot (-1)| = \frac{1}{6}.$$

8. [**5 marks**] Two horses start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed.

   Solution: Let S be the starting point and let F be the finishing point of the race. Let $f(t)$ and $g(t)$ denote the distances of the two horses from the starting point at time t. Suppose that the horses need time T to reach the point F. We take that f and g are continuous on $[0, T]$ and differentiable on $(0, T)$.

   It is given that

   $$f(0) = g(0) = 0 \text{ and } f(T) = g(T) = \text{ the distance from } S \text{ to } F.$$

   Let $h(t) = f(t) - g(t)$, for $t \in [0, T]$. Since f and g are continuous on $[0, T]$, it follows that h s continuous on $[0, T]$. Similarly, we conclude that h is differentiable on $(0, T)$. Next, we note that

   $$h(0) = f(0) - g(0) = 0 \text{ and } h(T) = f(T) - g(T) = 0.$$

   By Rolle's theorem it follows that there is $c \in (0, T)$ such that $h'(c) = 0$. Since $h'(c) = f'(c) - g'(c)$ we conclude that

   $$f'(c) = g'(c).$$

   Thus at time c both horses are with the same speed.

9. [**10 marks**] Sketch the graph of

   $$y = 4x^{1/3} + x^{4/3}.$$

   On your graph **clearly** indicate and label all intercepts, local extrema, and inflection points. **For full marks you have to show all your work.**

   Solution: Let $f(x) = 4x^{1/3} + x^{4/3}$.

   (a) The domain of f is the set of all real numbers.

   (b) From $f(0) = 0$ it follows that the origin is both an x and the y intercept. From

   $$f(x) = 4x^{1/3} + x^{4/3} = x^{1/3}(4 + x)$$

   we get that that $(-4, 0)$ is another x intercept.

(c) From

$$\lim_{x\to\pm\infty} f(x) = \infty$$

we conclude that there is no horizontal asymptote. Since f is continuous on its domain, there is no vertical asymptote. From

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^{1/3}(4+x)}{x} = \lim_{x\to\pm\infty} x^{1/3}\left(\frac{4}{x}+1\right) = \pm\infty$$

we conclude that there is no slant asymptote.

(d) From

$$f'(x) = \frac{4}{3}\cdot x^{-2/3} + \frac{4}{3}\cdot x^{1/3} = \frac{4}{3}\cdot x^{-2/3}(1+x) = \frac{4}{3}\cdot\frac{1+x}{\sqrt[3]{x^2}}$$

we conclude that $x=-1$ and $x=0$ are critical numbers.
Since, for $x \neq 0$, $\sqrt[3]{x^2} > 0$ we conclude that the sign of $f'(x)$ depends only on the sign of $1+x$. Thus:

x		-1		0	
$1+x$	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	undefined	$+$
$f(x)$	$\searrow$	local minimum	$\nearrow$	inflection point	$\nearrow$

Therefore f is increasing on $(-1,0)$ and $(0,\infty)$ and decreasing on $(-\infty,-1)$. By the first derivative test, there is a local minimum at the point $(-1,f(-1)) = (-1,-3)$ and an inflection point at $(0,f(0)) = (0,0)$. Also we see that there is a vertical tangent line at (0.0).

(e) From

$$f'(x) = \frac{4}{3}\cdot\left(x^{-2/3} + x^{1/3}\right)$$

it follows that

$$f\text{"}(x) = \frac{4}{3}\cdot\left(-\frac{2}{3}\cdot x^{-5/3} + \frac{1}{3}\cdot x^{-2/3}\right) = \frac{4}{9}\cdot x^{-5/3}(-2+x) = \frac{4}{9}\cdot\frac{x-2}{x\sqrt[3]{x^2}}.$$

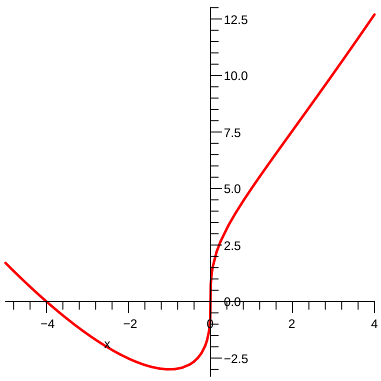

Figure 1: Problem 9

As before we conclude that the sign of $f"(x)$ depends only on the sign of $\frac{x-2}{x}$. Thus:

x		0		2	
$x-2$	$-$	$-$	$-$	0	$+$
x	$-$	0	$+$	$+$	$+$
$f"(x)$	$+$	undefined	$-$	0	$+$
$f(x)$	$\cup$	inflection point	$\cap$	inflection point	$\cup$

Therefore f is concave upward on $(-\infty, 0)$ and $(2, \infty)$ and concave downward on $(0, 2)$. It follows that $(0, 0)$ and $(2, f(2)) = (2, 6\sqrt[3]{2})$ are inflection points.

(f) Graph: See Figure 1.

10. [**5 marks**] An open-topped cylindrical pot is to have volume 250 cm^2. The material for the bottom of the pot costs 4 cents per cm^2; that for its curved side costs 2 cents per cm^2. What dimensions will minimize the total cost of this pot?

[**Note:** The area of a circle with the radius r equals $B = r^2\pi$; the circumference of the circle with the radius r equals $c = 2r\pi$; the volume of the cylinder is the product of the area of the base and the height.]

Solution: Let $r > 0$ be the radius of the bottom of the pot and let h be the height of the pot. Since the volume of the pot is is the product

of the area of the base and the height, it is given that

$$250 = r^2\pi h \Rightarrow h = \frac{250}{r^2\pi}.$$

The cost of the pot is given by

$C = 4 \cdot (\text{area of the bottom}) + 2 \cdot (\text{area of the side}) = 4r^2\pi + 2 \cdot 2r\pi h.$

Since $h = \frac{250}{r^2\pi}$, we get that the cost is given by

$$C = C(r) = 4r^2\pi + 4r\pi \cdot \frac{250}{r^2\pi} = 4r^2\pi + \frac{1000}{r}.$$

From

$$C' = 8r\pi - \frac{1000}{r^2}$$

we get that the critical numbers, for $r > 0$, are given by

$$8r\pi - \frac{1000}{r^2} = 0 \Leftrightarrow r = 5\pi^{-1/3}.$$

From $C" = 8\pi - \frac{2000}{r^3}$ and $C"(5\pi^{-1/3}) = 6\pi > 0$, by the second derivative test we conclude that C has a local minimum at $r = 5\pi^{-1/3}$.

Next, we observe that

$$\lim_{r\to 0} C(r) = \lim_{r\to 0} 4r^2\pi + \frac{1000}{r^2} = \infty$$

and

$$\lim_{r\to\ infty} C(r) = \lim_{r\to\infty} 4r^2\pi + \frac{1000}{r^2} = \infty.$$

Therefore C has the absolute minimum at $r = 5\pi^{-1/3}$.

The dimensions that will minimize the total cost are

$$r = 5\pi^{-1/3} \text{ and } h = 10\pi^{-1/3}$$

centimeters.

11. **[5 marks]**

(a) Show that Newton's method applied to the equation

$$\frac{1}{x} - a = 0$$

yields the iterative formula

$$x_{n+1} = 2x_n - a(x_n)^2$$

and thus provides a method for approximating the reciprocal $1/a$ without performing any divisions.

Solution: Let $f(x) = \frac{1}{x} - a$. Then $f'(x) = -\frac{1}{x^2}$. If x_n is the nth iteration obtained by Newton's method then

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} \\ &= x_n + \frac{\frac{1-ax_n}{x_n}}{-\frac{1}{x_n^2}} \\ &= x_n + x_n - ax_n^2 = 2x_n - a \end{aligned}$$

(b) Approximate $1/7$ by taking $x_0 = 0.12$ and calculating x_1.
[**Note:** Calculator gives $1/7 \approx 0.1428$.]
Solution: If $a = 7$ and $x_0 = 0.12$ then

$$x_1 = 2 \cdot 0.12 - 7 \cdot (0.12)^2 = 0.24 - 0.1008 = 0.1392.$$

12. [**5 marks**] A particle starts from rest (that is with initial velocity zero) at the point $x = 10$ and moves along the x-axis with acceleration function $a(t) = 12t$. Find the resulting position function $x(t)$.

Solution: It is given that $x(0) = 10$ and $x'(0) = v(0) = 0$. From $a(t) = 12t$ we get that the velocity, as an antiderivative of acceleration is given by

$$x'(t) = v(t) = 6t^2 + c$$

for some constant c. From $v(0) = c$ and the initial condition $v(0) = 0$ we conclude that $c = 0$. Thus $x'(t) = 6t^2$. This implies that there is a constant d such that

$$x(t) = 2t^3 + d.$$

From $x(0) = 10$ we get $d = 10$. Thus the resulting position function is

$$x(t) = 2t^3 + 10.$$

13. [**4 marks**] Sketch the curve

$$x = \sin^2 \pi t, \ y = \cos^2 \pi t, \ 0 \le t \le 2.$$

Clearly label the initial and terminal points and describe the motion of the point $(x(t), y(t))$ as t varies in the given interval.

Solution: We note the following two facts.

- For all $t \in [0, 2]$, $0 \le x \le 1$ and $0 \le y \le 1$.
- For all $t \in [0, 2]$, $x + y = 1$.

Thus, the points from the given curve belong to the line segment of the line $x + y = 1$ that is in the first quadrant, together with the x and y intercepts.

Next we observe that $x(0) = 0$, $y(0) = 1$, $x(2) = 0$, and $y(2) = 1$. Thus the initial and the terminal points are at the point $(0, 1)$.

From $x(1/2) = 1$ and $y(1/2) = 0$ and the fact that x and y are continuous functions of t we conclude that the given curve is the line segment between the point $(0, 1)$ and the point $(1, 0)$.

The motion of the point is given by the following table:

t	0	$\rightarrow$	$\frac{1}{2}$	$\rightarrow$	1	$\rightarrow$	$\frac{3}{2}$	$\rightarrow$	2
$(x(t), y(t))$	$(0, 1)$	$\rightarrow$	$(1, 0)$	$\rightarrow$	$(0, 1)$	$\rightarrow$	$(1, 0)$	$\rightarrow$	$(0, 1)$

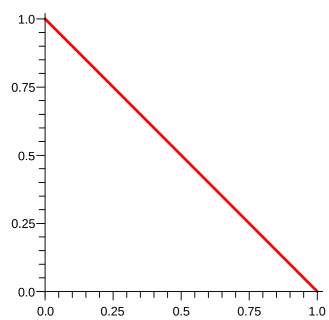

Figure 2: Problem 13