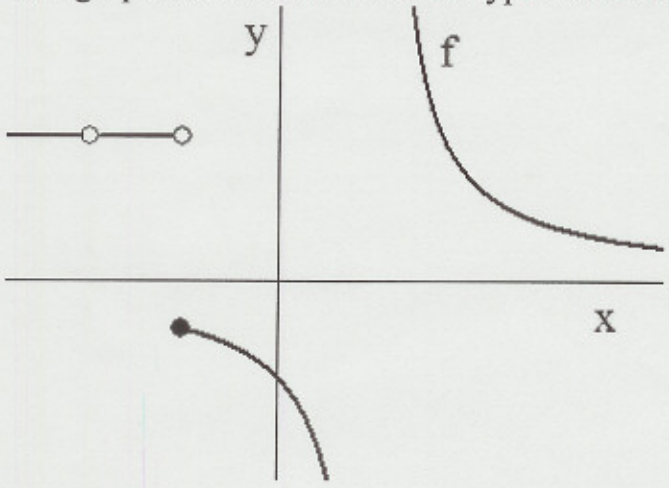
1. Answer **T** (true) or **F** (false) in the boxes provided or leave the box blank. No explanation is necessary. Every correct answer will receive **1**, no penalties for incorrect answers. **[10 marks]**

a) [F] The slope of the tangent line of $f(x)$ at the point $(a, f(a))$ is given by $\dfrac{f(a+h)-f(a)}{h}$.

b) [F] Using the Intermediate Value Theorem it can be shown that $\lim_{x\to 0} x\sin\dfrac{1}{x}=0$.

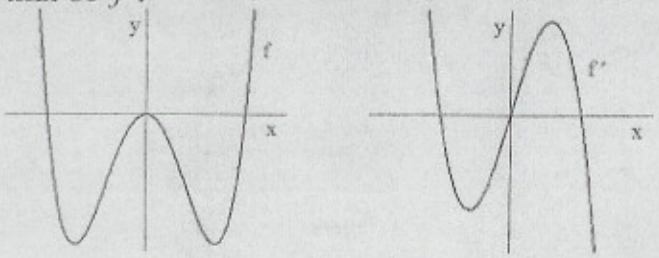
c) [T] The graph below exhibits three types of discontinuities.



d) [T] If $w=f(x), x=g(y), y=h(z)$ then $\dfrac{dw}{dz}=\dfrac{dw}{dx}\cdot\dfrac{dx}{dy}\cdot\dfrac{dy}{dz}$.

e) [T] Suppose that on the open interval I, f is a differentiable function that has an inverse function f^{-1} and $f'(x)\neq 0$. Then f^{-1} is differentiable and $\left[f^{-1}(x)\right]'=\dfrac{1}{f'(f^{-1}(x))}$ for all x in the domain of f^{-1}.

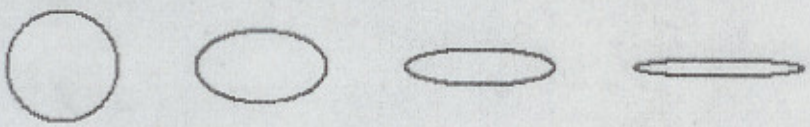
f) [F] Given the graph of f below to the left, the graph to the right must be that of f'.



g) [T] The conclusion of the Mean Value Theorem says that the graph of f has at least one tangent line in its interval I, whose slope is equal to the average slope on I.

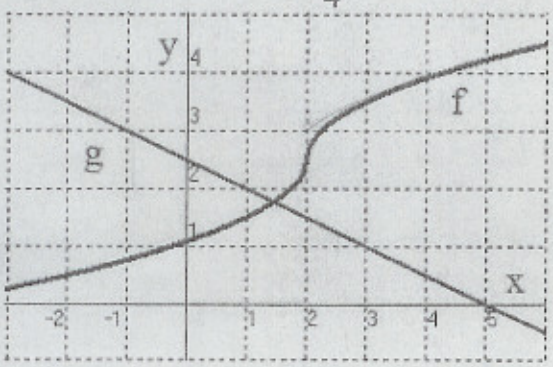
h) [F] The linear approximation $L(x)$ of a function $f(x)$ near the point $x=a$ is given by $L(x)=f'(a)+f(a)(x-a)$.

i) [F] The graphs below are labeled correctly with possible eccentricities for the given conic sections.



e=1 e=0.8 e=0.5 e=0.1

j) [T] Given $h(x)=g(f(x))$ and the graphs of f and g below then a good estimate for $h'(3)$ is $-\frac{1}{4}$.



2. Find the following limits: **[11 marks total]**

a) $\lim_{\theta \to \frac{\pi}{2}} \dfrac{\ln(\sin\theta)}{\cos\theta}$ **[4 marks]**

$$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{\theta \to \frac{\pi}{2}} \frac{\frac{\cos\theta}{\sin\theta}}{-\sin\theta} = \lim_{\theta \to \frac{\pi}{2}} \left(-\frac{\cos\theta}{\sin^2\theta}\right) = -\frac{0}{1^2} = 0$$

b) $\lim_{x \to \infty} \dfrac{1+3x}{\sqrt{2x^2+x}}$ **[4 marks]**

$$= \lim_{x \to \infty} \frac{\frac{1}{x} + 3}{\sqrt{2 + \frac{1}{x}}} = \frac{3}{\sqrt{2}}$$

c) Let $F(x)=\dfrac{2x^2-3x}{|2x-3|}$. **[3 marls]**

$$|2x-3| = \begin{cases} 2x-3, & x \geq \frac{3}{2} \\ -(2x-3), & x < \frac{3}{2} \end{cases}$$

i) Find $\lim\limits_{x\to 1.5^-} F(x)$.

$$\lim_{x\to 1.5^-} F(x) = \lim_{x\to 1.5^-} \frac{x(2x-3)}{-(2x-3)} = \lim_{x\to 1.5^-} (-x) = -1.5$$

ii) Find $\lim\limits_{x\to 1.5^+} F(x)$.

$$\lim_{x\to 1.5^+} F(x) = \lim_{x\to 1.5^+} \frac{x(2x-3)}{2x-3} = \lim_{x\to 1.5^+} x = 1.5$$

iii) Does $\lim\limits_{x\to 1.5} F(x)$ exist? Provide a reason.

$$\lim_{x\to 1.5^-} F(x) \neq \lim_{x\to 1.5^+} F(x),$$

therefore $\lim\limits_{x\to 1.5} F(x)$ DNE.

3. Differentiation questions. **[4 marks each]**

a) Given $y = \tan(\cos^{-1}(e^{4x}))$, find $\frac{dy}{dx}$. Do not simplify your answer.

$$\frac{dy}{dx} = \sec^2(\cos^{-1}(e^{4x})) \cdot \left(\frac{-1}{\sqrt{1-(e^{4x})^2}}\right) \cdot e^{4x} \cdot 4$$

b) Given $y = \frac{1}{x} + \cos 2x$, find $\frac{d^5y}{dx^5}$. Simplify your answer.

$$= x^{-1} + \cos 2x$$

$$\frac{dy}{dx} = -x^{-2} - 2\sin 2x$$

$$\frac{d^2y}{dx^2} = 2x^{-3} - 4\cos 2x$$

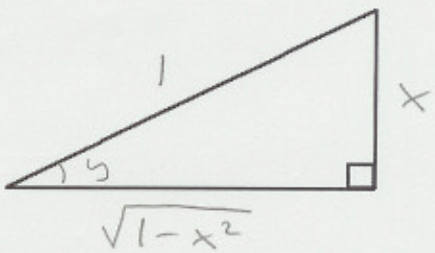
$$\frac{d^3y}{dx^2} = -6x^{-4} + 8\sin 2x$$

$$\frac{d^4y}{dx^2} = 24x^{-5} + 16\cos 2x$$

$$\frac{d^5y}{dx^5} = -120x^{-6} - 32\sin 2x$$

Show

c) Proof that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$. Label and use the right triangle below.



Let $y = \sin^{-1} x$,

then $x = \sin y$.

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\frac{\sqrt{1-x^2}}{1}} = \frac{1}{\sqrt{1-x^2}}$$

d) Suppose f is a differentiable function such that $f(g(x)) = x$, and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = \frac{1}{1+x^2}$.

$f'(x) > 0$ for all x,

So, $g'(x) = \frac{1}{f'(g(x))}$

$= \frac{1}{1+[f(g(x))]^2}$

$= \frac{1}{1+x^2}$.

$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(x)$

$f'(g(x))\, g'(x) = 1$

$g'(x) = \frac{1}{f'(g(x))}$

$= \frac{1}{1+[f(g(x))]^2}$

$= \frac{1}{1+x^2}$

e) Given $y=\dfrac{\sqrt{1+2x}\sqrt[4]{1+4x}\sqrt[6]{1+6x}\cdots\sqrt[100]{1+100x}}{\sqrt[3]{1+3x}\sqrt[5]{1+5x}\sqrt[7]{1+7x}\cdots\sqrt[101]{1+101x}}$, find y' at $x=0$.

$$\ln y = \tfrac{1}{2}\ln(1+2x) + \tfrac{1}{4}\ln(1+4x) + \ldots + \tfrac{1}{100}\ln(1+100x)$$
$$- \tfrac{1}{3}\ln(1+3x) - \tfrac{1}{5}\ln(1+5x) - \ldots - \tfrac{1}{101}\ln(1+101x)$$

differentiate both sides w.r.t. x:

$$\frac{1}{y}y' = \frac{2}{2(1+2x)} + \frac{4}{4(1+4x)} + \cdots + \frac{100}{100(1+100x)}$$
$$- \frac{3}{3(1+3x)} - \frac{5}{5(1+5x)} - \cdots - \frac{101}{101(1+101x)}$$

$$y'\Big|_{x=0} = \underbrace{y(0)}_{=1}\Big[\underbrace{1+1+\ldots+1}_{100\div 2=50 \text{ terms}} \underbrace{-1-1-\ldots-1}_{(101-1)\div 2=50 \text{ terms}}\Big]$$

$$= 1\cdot 0$$

$$= 0$$