Math 151 Final 2008

12

(1) [Marks: 4] Use the limit laws to find the following limits or explain why they don't exist.

(a) $\lim_{x\to 3} \frac{\sqrt{3x} - 3}{|3-x|}$ $\times \frac{\sqrt{3x}+3}{\sqrt{3x}+3} = \frac{3x-9}{|3-x|(\sqrt{3x}+3)}$

$= \frac{3(x-3)}{|3-x|(\sqrt{3x}+3)}$;

$\lim_{x\to 3^-} = \frac{3(x-3)}{3-x(\sqrt{3x}+3)} \Rightarrow -\frac{3}{6} = -\frac{1}{2}$

$\lim_{x\to 3^+} = \frac{3(x-3)}{x-3(\sqrt{3x}+3)} \to \frac{3}{6} = \frac{1}{2}$

NOT exist ✓

(b) $\lim_{x\to\infty} x\tan\left(\frac{2}{3x}\right)$ form $\infty \cdot 0$. L'Hop:

$\frac{\tan\left(\frac{2}{3x}\right)}{1/x} \rightsquigarrow \frac{\frac{2}{3}\sec^2\left(\frac{2}{3x}\right)\left(-\frac{1}{x^2}\right)}{\left(-1/x^2\right)}$

$= \frac{2}{3}\sec^2\left(\frac{2}{3x}\right) \to \frac{2}{3}\sec^2(0) = \frac{2}{3}$ ✓

(c) $\lim_{x\to 0^+} 2x^{\sqrt{x}}$ $\quad y = 2x^{\sqrt{x}}$

$\ln y = \ln 2 + \sqrt{x} \ln x$

$\lim_{x\to 0^+} \ln y = \ln 2 + \underbrace{\lim_{x\to 0^+} \sqrt{x} \ln x}_{0\cdot\infty}$

$\to \frac{\ln x}{1/\sqrt{x}} \xrightarrow{L'H} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = -2x^{\frac{1}{2}} \to 0$

So $\lim \ln y = \ln 2 + 0$

$\to y \to e^{\ln 2} = 2$ ✓

(d) $\lim_{x\to \frac{\pi}{4}} \frac{2\ln(\tan x)}{3\sin(4x)} \sim \frac{0}{0}$

L'H; $\frac{2}{3} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{4\cos 4x} = \frac{1}{6} \frac{\sin x / \cos^3 x}{\cos 4x}$

$\to \frac{1}{6} \frac{1\cdot 2}{-1} = -\frac{2}{6} = -\frac{1}{3}$ ✓

12

(2) [Marks: 4] Find the derivatives of the following functions.

(a) $f(x) = 1 - 3^x + x^3$

$$f' = -\ln 3 \cdot 3^x + 3x^2$$

(b) $g(z) = \sqrt{\arctan(\sqrt{z})}$

$$g' = \frac{\frac{1}{2} \cdot \frac{1}{1+z} \cdot \frac{1/2}{\sqrt{z}}}{\sqrt{\arctan(\sqrt{z})}}$$

$$= \frac{1/4}{(1+z)\sqrt{z}\,\sqrt{\arctan(\sqrt{z})}}$$

✓

(c) $y = x - \log_2(\sqrt{x^2+2})\cos(2x)$

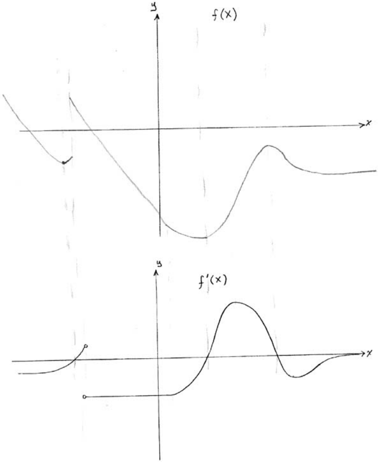
$$= 1 - \frac{\frac{x}{\sqrt{x^2+2}}}{\ln 2\left(\sqrt{x^2+2}\right)} \cdot \cos(2x) + 2\log_2\left(\sqrt{x^2+2}\right)\sin(2x)$$

(d) $h(t) = \sqrt{\dfrac{2t^2+1}{t^2-2}}$

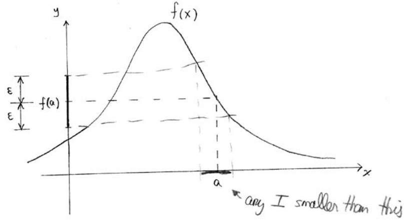
$$h' = \frac{1}{2}\left(\frac{2t^2+1}{t^2-2}\right)^{-1/2} \cdot \left[\frac{(4t)(t^2-2) - (2t^2+1)(2t)}{(t^2-2)^2}\right]$$

6

(3) [Marks: ~~4~~] Below is the graph of $f'(x)$, the <u>derivative</u> of a function $f(x)$. Above it sketch the graph of $f(x)$. (Did you read this question carefully?)



(4) [Marks: ~~4~~ 8] (a) Here is a graph of a function $f(x)$ along with an interval of length 2ε centred at the point $f(a) = L$. Sketch an interval I around a such that if $x \in I$ then $|f(x) - L| < \varepsilon$.



(b) Use the precise $\varepsilon - \delta$ method to prove that

$$\lim_{x \to 1} \frac{x^2 + 1}{2x} = 1$$

$$|f(x) - L| = \left| \frac{x^2 + 1 - 2x}{2x} \right| = |x - 1|^2 \left| \frac{1}{2x} \right|$$

$$< \delta^2 \left| \frac{1}{2x} \right|$$

Now suppose $|x - 1| < \frac{1}{2}$; then $\frac{1}{2x} < 1$, so

$|f(x) - L| < \delta^2 \cdot 1$. So if $\delta = \sqrt{\varepsilon}$ ✓

answer: $\delta = \min\left\{ \sqrt{\varepsilon}, \frac{1}{2} \right\}$

(5) [Marks: ~~4~~ 6] Find values of the constants a and b so that the following function is differentiable everywhere.

$$f(x) = \begin{cases} ax^2 + bx & x < 1 \\ (x-2)^2 + 3 & x \geq 1 \end{cases}$$

$x^2 - 4x + 9$

Illustrate your answer by sketching the resulting $f(x)$.

Continuous @ $x = 1$;

$$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= a(1) + b(1) = a+b \\ \lim_{x \to 1^+} f(x) &= (1-2)^2 + 3 = 4 \end{aligned}\right\} \quad a+b = 4$$

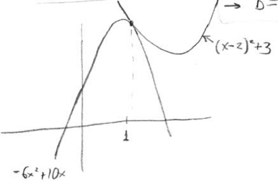
Diff'ble @ $x = 1$;

$$\left.\begin{aligned} \lim_{h \to 0^-} \frac{f(1) - f(1+h)}{h} &= 2ax+b \Big|_{x=1} = 2a+b \\ \lim_{h \to 0^+} \frac{f(1) - f(1+h)}{h} &= 2x-4 \Big|_{x=1} = -2 \end{aligned}\right\} \quad 2a+b = -2$$

Solve: $b = 4 - a$;

$2a + b = 2a + 4 - a = -2 \rightarrow a = -6$

$\rightarrow b = 10$



$-6x^2 + 10x$

$6\left(x^2 - \frac{5}{3}x\right)$

$-6\left[\left(x - \frac{5}{6}\right)^2 \mp \frac{25}{36}\right]$

$-6\left(x - \frac{5}{6}\right)^2 + \frac{25}{6}$

6

(6) [Marks: 4] (a) State the Mean Value Theorem. Make a sketch illustrating the theorem.

If f cont on $[a,b]$, diff'ble on (a,b), then there is a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Use the Mean Value Theorem to prove that the equation $2x^4 + x = 3$ has at most two solutions.

$f' = 8x^3+1$ has one root $\left(-\frac{1}{8}\right)^{\frac{1}{3}} = -\frac{1}{2}$ (*)

Suppose there were 3 roots; x_1, x_2, x_3. Then for intervals $[x_1,x_2]$, $[x_2,x_3]$ MVT $\Rightarrow$

$f'(c_1) = 0 = f'(c_2)$ with $c_1 \neq c_2$

This contradicts (*)

6

(7) [Marks: ~~4~~] Find the slant asymptote of the function $f(x) = 5x + 2 - \sqrt{9x^2 + x + 1}$ as $x \to +\infty$ (only this one limit).

Find m;

$$\lim_{x \to +\infty} f' = \lim_{x \to +\infty} \left(5 - \frac{9x + 1/2}{\sqrt{9x^2+x+1}} \right)$$

$$= \lim \left(5 - \frac{x\left(9 + \frac{1}{2x}\right)}{x\sqrt{9 + \frac{1}{x} + \frac{1}{x^2}}} \right) = 5 - \frac{9}{\sqrt{9}} = 5 - 3 = \underline{2}$$

Find b;

$$\lim_{x \to \infty} \left(f(x) - 2x \right) = \lim_{x \to \infty} \left(5x + 2 - \sqrt{9x^2+x+1} - 2x \right)$$

$$= \lim \left(3x + 2 - \sqrt{9x^2+x+1} \right) \times \frac{(3x+2) + \sqrt{9x^2+x+1}}{(3x+2) + \sqrt{9x^2+x+1}}$$

$$= \lim \frac{\overset{9x^2+12x+4}{(3x+2)^2} - (9x^2+x+1)}{3x+2+\sqrt{9x^2+x+1}} = \lim \frac{11x+3}{x\left(3 + \frac{2}{x} + \sqrt{9 + \frac{1}{x} + \frac{1}{x^2}}\right)}$$

$$= \frac{11}{3+3} = \frac{11}{6}$$

so $\boxed{y = 2x + \frac{11}{6}}$ ✓

$x(2x-3)$; $x=0, \frac{3}{2}$

12

(8) [Marks: ~~4~~] Make a sketch of the graph of $f(x) = \dfrac{2x^2-3x}{(x+5)^2}$ by following the steps below. Here are the first and second derivatives of $f(x)$;

$$f'(x) = \frac{23x-15}{(x+5)^3}, \quad f''(x) = \frac{-46x+160}{(x+5)^4}$$

(a) What is the domain of $f(x)$?

$x \neq -5$

(a) Find the critical points of $f(x)$ and classify them using the first derivative test.

$23x-15=0 \rightarrow \boxed{x = \frac{15}{23}}$ f' changes from -ve to +ve → loc min

$\boxed{x=-5}$ Not local extrema

(b) Find regions where $f(x)$ is increasing or decreasing.

$f'>0$; $x<-5$, $x > \frac{15}{23} \sim .65$

$f'<0$; $-5 < x < \frac{15}{23}$

(c) Find any asymptotes. Be sure to determine how the graph of $f(x)$ approaches the asymptote.

vert @ $x = -5$;

$\lim_{x \to -5^-} f = +\infty$

$\lim_{x \to -5^+} f = -\infty$

horizontal: $\lim_{x \to +\infty} f = \lim \frac{x^2\left(2 - \frac{3}{x}\right)}{x^2\left(1 + \frac{5}{x}\right)^2} = 2$

$f(x) \sim \frac{2-\varepsilon}{(1+\varepsilon)^2}$ so $f < 2$ (below)

$\lim_{x \to -\infty} f = 2$; $f(x) \sim \frac{2+\varepsilon}{(1-\varepsilon)^2}$ so $f > 2$ (above)

(d) Find any points of inflection and determine regions where the graph is concave up or concave down.

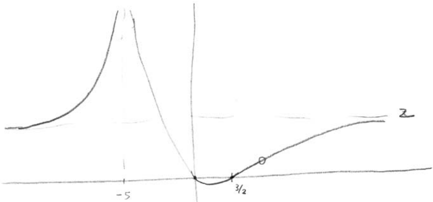
$f'' = 0$; $x = 160/46 \sim 3.5 \to \left(\frac{160}{46}, f\left(\frac{160}{46}\right)\right)$

CU: $x < -5$, $-5 < x < \frac{160}{46}$

CD: $x > 160/46$

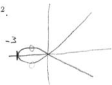
f'' not defined at $x = -5$ but NOT a point of inflection

(e) Use this information to make a sketch of $f(x)$ below.



9

(9) [Marks: 4] Consider the curve defined implicity by the equation $y^2 = x^3 + 3x^2$.



(a) Find two points on the curve where the tangent line is horizontal.

4

y': $\quad 2yy' = 3x^2 + 6x$

$\rightarrow y' = \dfrac{3x^2+6x}{2y} = \dfrac{3x(x+2)}{2y}$

$y' = 0$; $\quad x = 0, -2$

$\rightarrow y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$

$\rightarrow$ two points: $(-2, 2)$, $(-2, -2)$

(b) Find a point on the curve, but <u>not</u> the origin $(0, 0)$, where the tangent line is vertical.

2

need $y' = \pm\infty$; $\quad$ ie $y = 0$; $\quad x^3 + 3x^2 = 0$

$x^2(x+3) = 0$

$\boxed{x = -3}$

$(-3, 0)$

(b) Is the curve concave up or concave down at the point $(1, 2)$?

3

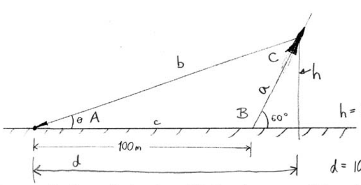
y''; $\quad y'' = \dfrac{(6x+6)(2y) - (3x^2+6x)2y'}{(2y)^2}$

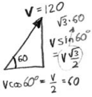
@ $(1,2)$; $\quad y' = \dfrac{3(1)(1+2)}{2 \cdot 2} = \dfrac{3 \cdot 3}{4} = \dfrac{6}{4} = \dfrac{3}{2}$

$\rightarrow y'' = \dfrac{(6+6)(4) - (3+6)(3)}{16} > 0 \quad$ so CU

6

(10) [Marks: 4] A video camera on the ground is following a rocket that has been launched 100 metres away. The rocket is following a straight line that is inclined 60^o from the horizontal at a speed of 120 metres per second (see diagram below).





$h = \frac{\sqrt{3}}{2} v \cdot t = \sqrt{3} \cdot 60 t$

$d = 100 + 60t$

Find an expression that would tell you how quickly (in radians per second) the angle θ of elevation of the camera is changing 2 minutes after launch. Do not calculate this number: only provide the formula that would allow you to do so. You must also show how all quantities appearing in your formula would be calculated. You may need (some of) these identities:

$$\sin^2 x + \cos^2 x = 1, \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \quad \boxed{a^2 = b^2 + c^2 - 2bc \cos A}$$

$2aa' = 2bb' - 2b'c \cos A + 2bc (\sin A) A'$

$\rightarrow \boxed{A' = \dfrac{aa' - bb' + b'c \cos A}{bc \sin A}}$

angle A: @ $t = 2\text{min} = T$ $\quad \tan A = \frac{h}{d} = \frac{\sqrt{3} \cdot 60 \cdot 120}{100 + 60 \cdot 120} \rightarrow A = \tan^{-1}(\cdots)$

or just $\boxed{\cos A = \frac{d}{b}} = \frac{d}{\sqrt{d^2 + h^2}}$; $\boxed{\sin A = \frac{h}{b}}$ $\quad h = \sqrt{3} \cdot 60 \cdot 120$

$a' = 120$,

b' ; $\quad b^2 = d^2 + h^2 \rightarrow 2bb' = 2dd' + 2hh' \rightarrow b' = \frac{dd' + hh'}{b}$

$d' = 60$, $h' = \sqrt{3} \cdot 60$

$d = 100 + 60 \cdot 120$

$h = \sqrt{3} \cdot 60 \cdot 120$

(the next page is blank if you need it for your solution)

(11) [Marks: 4] Find the absolute maximum and absolute minimum values of $f(x) = x^2 - \ln x$ on the interval $[\frac{1}{2}, 2]$.

$f'(x) = 2x - \frac{1}{x}$ crit. pts: $2x^2 = 1 \rightarrow x = \frac{+1}{\sqrt{2}}$ (just one)

$f(\frac{1}{2}) = \frac{1}{4} - \ln\frac{1}{2} = 0.943$

$f(\frac{1}{\sqrt{2}}) = 0.846$ ← min

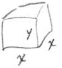
$f(2) = 4 - \ln 2 = 3.306$ ← max

$\boxed{\text{By EVT}}$, abs min @ $\frac{1}{\sqrt{2}}$; value $\underline{0.846}$

abs max @ 2 ; value $\underline{3.306}$

(12) [Marks: 4] A closed box with a square base is to be built. The cost of the material for the 4 sides is \$20 per square metre and the cost of the material for the top and bottom is \$50 per square metre.

If you have \$100 to spend for material to build the box, what is the maximum volume of box that you can build?



Cost $= 20(4xy) + 50(2x^2) = 100$

$\rightarrow y = \dfrac{5 - 5x^2}{4x}$

Volume: $x^2 y = \dfrac{5x - 5x^3}{4} = \frac{5}{4} x (1 - x^2)$

$0 \le x \le 1$

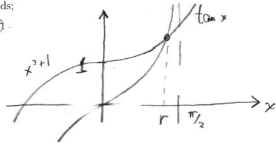
$V'(x) = \frac{5}{4}(1 - 3x^2) = 0$; $x = \frac{1}{\sqrt{3}}$

$V'' = -\frac{5}{2} 6 \cdot x < 0$; local max.

So, $x = \frac{1}{\sqrt{3}}$, $\rightarrow V = \frac{5}{4}\sqrt{3}(1 - \frac{1}{3}) = \frac{5\sqrt{3}}{6}$

(13) [Marks: ~~4~~ 7] Make a sketch that shows that the graphs of $\tan x$ and x^3+1 intersect at some positive x-value. Then find an approximation to this value by the following methods;

1 skt.



(a) Use the bisection method beginning with the interval $[0, 1.5]$ to find an interval of length less than 0.5 that contains a solution.

3

$f(x) = \tan x - x^3 - 1$

$f(0) = -1$, $f(1.5) > 0$ ✓ $\rightarrow$ root in $[0, 1.5]$ by IVT

$x_1 = \frac{3}{4}$; $f(\frac{3}{4}) < 0 \rightarrow [\frac{3}{4}, \frac{3}{2}]$ $\leftarrow \frac{3}{4} \rightarrow$

$\rightarrow X_2 = \frac{9}{8}$

$f(\frac{9}{8}) < 0 \rightarrow [\frac{9}{8}, \frac{3}{2}]$ ✓

$\leftarrow \frac{3}{8} \rightarrow$

(b) Apply Newton's method for one iteration starting with your approximation in part (a). (If you didn't get (a), use 1 as a starting point.)

3

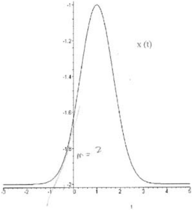
$f' = \sec^2 x - 3x^2$

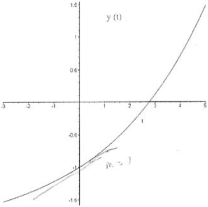
$X_1 = 1.3125$

$X_2 = 1.3121 - \frac{f(1.3125)}{f'(1.3125)} = 1.3125 - \frac{.524}{10.158}$

$= 1.261$ ✓

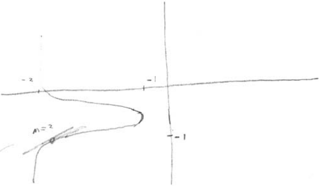
(14) [Marks: ~~4~~ 9] Consider functions $x(t)$ and $y(t)$ given below;





(a) Make a sketch of the parametric curve $(x(t), y(t))$.

6



(b) Using the graphs of $x(t)$ and $y(t)$ (not any formula for $x(t), y(t)$ you may guess at), find a point on the parametric curve where the tangent line has slope 2. (Hint: Remember the formula for dy/dx.) Explain your reasoning by referring to the graphs.

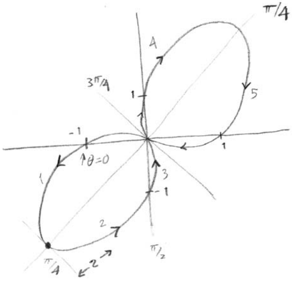
$\frac{dy}{dx} = \frac{y'}{x'} = 2$; need $y' \sim 2x'$ at same t

3 (ie, slope $y(t) \sim 2 \cdot$ slope $x(t)$)

$t \sim -1 \rightarrow (x, y) \sim (-1.8, -1.2)$

(15) [Marks: 4] (a) Make a sketch of the polar curve $r = \cos\left(2\overset{\theta}{t} + \frac{\pi}{2}\right) - 1$. Be sure to indicate directions which determine special features of the graph, the size of the graph, and any points where the curve crosses the x and y axis.

6



$\theta = \frac{\pi}{4} \rightarrow 2\theta = \frac{\pi}{2} \rightarrow \cos\left(2\theta + \frac{\pi}{2}\right) = \cos(\pi) = -1$

$\theta = \frac{\pi}{2} \rightarrow 2\theta = \pi \rightarrow r = \cos\left(\frac{3\pi}{2}\right) - 1 = -1$

$\theta = \frac{3\pi}{4} \rightarrow r = \cos\left(\frac{3\pi}{2} + \frac{\pi}{2}\right) - 1 = 0$ ✓

$r = 0 \rightarrow \cos\left(2\theta + \frac{\pi}{2}\right) = 1 \rightarrow 2\theta + \frac{\pi}{2} = 2n\pi \rightarrow \theta = \frac{2n\pi - \frac{\pi}{2}}{2} = n\pi - \frac{\pi}{4} = \frac{3\pi}{4}$