(1) [Marks: 12] Find the limits, or explain why they don't exist.

(a) $\lim_{x \to 1} |x-1| \sin\left(\frac{2}{x-1}\right)$

$\lim_{x \to 1} |x-1| = 0$; $\left| \sin\left(\frac{2}{x-1}\right) \right| \le 1, \quad x \ne 1$

squeeze theorem $\Rightarrow$ lim $= 0$

(b) $\lim_{x \to 3} \frac{2x^2 - 2x - 12}{4|3-x|}$

$= \lim \frac{(x-3)(x+2)}{2|3-x|} = \frac{5}{2} \lim_{x \to 3} \frac{(x-3)}{|3-x|} = \begin{cases} +\frac{5}{2} & x \to 3^+ \\ -\frac{5}{2} & x \to 3^- \end{cases}$

one-sided limits unequal $\Rightarrow$ limit does not exist

(c) $\lim_{x\to\infty} \frac{\log_2(x^3+2)}{\sqrt{x-1}}$; $\frac{\infty}{\infty}$ use L'Hospital's Rule

$$= \lim_{x\to\infty} \frac{\frac{1}{\ln 2}\cdot\frac{3x^2}{x^3+2}}{\frac{1}{2\sqrt{x-1}}} = \frac{6x^2\sqrt{x-1}}{\ln 2 \;\; x^3+2} \;;$$

$$= \lim_{x\to\infty} \frac{6x^{5/2}\sqrt{1-\frac{1}{x}}}{\ln 2\, x^3\left(1+\frac{2}{x^3}\right)} = \lim_{x\to\infty} \frac{6\sqrt{1-\frac{1}{x}}}{\ln 2\, x^{1/2}\left(1+\frac{2}{x^3}\right)} = 0$$

(d) $\lim_{x\to 0} (1+\tan x)^{1/x}$ $\qquad y = (1+\tan x)^{1/x}$

$\ln y = \frac{1}{x}\ln(1+\tan x)$; form $\infty\cdot 0$. Re-write as $\frac{0}{0}$;

$\lim_{x\to 0} \frac{\ln(1+\tan x)}{x}$; L'Hospital $\rightarrow$

$$\lim_{x\to 0} \frac{\frac{1}{1+\tan x}\cdot\frac{1}{1+x^2}}{1} = 1.$$

So $\lim_{x\to 0} y = e^1 = e$

(2) [Marks: 16] Find the indicated derivatives. You do not need to simplify your answer.

(a) z'; $z = \left(u + \dfrac{1}{u-1}\right)^{-5/3}$

$$z' = -\frac{5}{3}\left(u + \frac{1}{u-1}\right)^{-8/3}\left(1 - \frac{1}{(u-1)^2}\right)$$

(b) y'; $y = \sinh(\cos\left(\dfrac{1}{x^2}\right))$

$$y' = \cosh\left(\cos\left(\frac{1}{x^2}\right)\right)\cdot\left(-\sin\left(\frac{1}{x^2}\right)\right)\cdot\left(\frac{-2}{x^3}\right)$$

(c) y''; $x^3 - y^2 + y^3 = x$

$$3x^2 - 2yy' + 3y^2y' = 1 \quad \rightarrow \quad y' = \frac{1-3x^2}{3y^2-2y}$$

$$\rightarrow y'' = \frac{(-6x)(3y^2-2y) - (1-3x^2)(6yy'-2y')}{(3y^2-2y)}$$

$$= \frac{(-6x)(3y^2-2y) - (1-3x^2)\left[2\left(\frac{1-3x^2}{3y^2-2y}\right)(3y-1)\right]}{(3y^2-2y)}$$

(d) $\dfrac{dh}{dt}$; $h(t) = t^s - s^t$ $\quad (s > 0)$

$$h' = s t^{s-1} - (\ln s) s^t ,$$

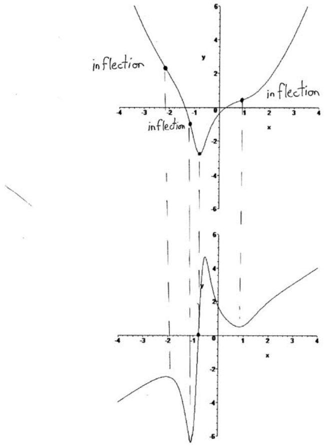
(e) $f'(x)$; $f(x) = (1/x)^{\ln x}$

$$\ln f(x) = \ln x \ln\left(\frac{1}{x}\right)$$

$$\rightarrow \frac{f'}{f} = \frac{1}{x} \ln\left(\frac{1}{x}\right) + \ln x \underbrace{\left(x \cdot \frac{-1}{x^2}\right)}_{-\frac{1}{x}}$$

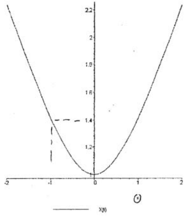
$$\rightarrow f'(x) = \left(\frac{1}{x}\right)^{\ln x} \frac{1}{x} \Bigg[\underbrace{\ln\left(\frac{1}{x}\right) - \ln x}_{-2 \ln x} \Bigg]$$

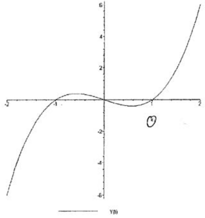
(3) [Marks: 6] Below is the graph of a function $f(x)$. In the space below it, sketch the derivative $f'(x)$. You should make this as accurate as possible. Identify any points of inflection of $f(x)$.

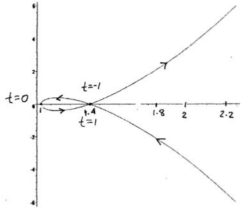


1

(4) [Marks: 10] Below are the graphs of $x(t)$ and $y(t)$. In the space below, sketch the parametric curve $(x(t), y(t))$. Indicate which direction along the curve corresponds to increasing parameter t. Your sketch should be made as accurate as possible. $t=-1,0,1$

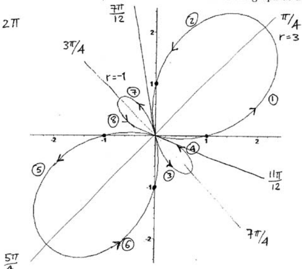






(5) [Marks: 12] (a) Sketch the following polar graph; $r = 2\sin(2\theta) + 1$. Be sure to label any angles that determine any special features of the graph, locations where the graph crosses the axis, and to indicate the size of the graph as accurately as you can.

$\theta : 0 \to 2\pi$



Indicate directions of increasing θ with arrow

& origin

x intercepts ; ± 1 $(\theta = 0, 2\pi)$

y intercepts ; ± 1 $(\theta = \frac{\pi}{2}, \frac{3\pi}{2})$

r max @ $\theta = \frac{\pi}{4}$ $(r=3)$

r max @ $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ $(r=-3)$

$r=0$; $\sin 2\theta = -\frac{1}{2}$; $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$

(b) Find the equation of the tangent line to the curve at the point on the curve where $\theta = \pi/6$.

$$r = f(\theta) = 2\sin(2\theta) + 1$$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{(4\cos 2\theta)(\sin\theta) + (2\sin 2\theta + 1)\cos\theta}{(4\cos 2\theta)(\cos\theta) - (2\sin 2\theta + 1)\sin\theta}$$

$$\theta = \frac{\pi}{6} \; ; \quad \frac{dy}{dx} = \frac{\left(4\cdot\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(2\cdot\frac{\sqrt{3}}{2} + 1\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(4\cdot\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(2\,\frac{\sqrt{3}}{2} + 1\right)\left(\frac{1}{2}\right)} = \frac{1 + \frac{(\sqrt{3}+1)\sqrt{3}}{2}}{\sqrt{3} - \frac{\sqrt{3}+1}{2}} = m \quad \sim 9.2$$

Point: $(r, \frac{\pi}{6}) = \left(2\sin\frac{\pi}{3} + 1, \frac{\pi}{6}\right) = \left(\sqrt{3}+1, \frac{\pi}{6}\right) \approx \left(2.73, \frac{\pi}{6}\right)$

Cartesian: $(x, y) = \left(r\cos\frac{\pi}{6}, r\sin\frac{\pi}{6}\right) = \left((\sqrt{3}+1)\frac{\sqrt{3}}{2}, (\sqrt{3}+1)\frac{1}{2}\right) \approx (2.37, 1.37) = (c, d)$

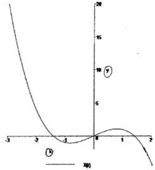
equation of tangent: $y = d + m(x - c)$

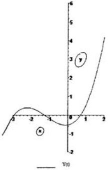
$= mx + (d - mc)$

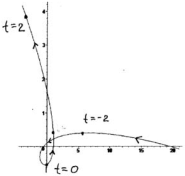
$\approx 9.2x - 20.4$

2

(4) [Marks: 10] Below are the graphs of $x(t)$ and $y(t)$. In the space below, sketch the parametric curve $(x(t), y(t))$. Indicate which direction along the curve corresponds to increasing parameter t. Your sketch should be made as accurate as possible. $t = -2, 0, 2$



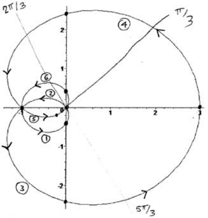




(5) [Marks: 12] (a) Sketch the following polar graph; $r = 1 - 2\cos(\theta/2)$. Be sure to label any angles that determine any special features of the graph, locations where the graph crosses the axis, and to indicate the size of the graph as accurately as you can.

θ: $0 \to 4\pi$

$r=-1$ @ $\theta=0$



Directions of increasing θ.

x intercepts: $-1, 3, 0$ $(\theta = 0, \pi)$

y intercepts: $1-\sqrt{2}$, $1+\sqrt{2}$ $\left(\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}\right)$

r max; 3; $\theta = 2\pi$

$r=0$; $\cos\frac{\theta}{2} = \frac{1}{2}$; $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(b) Find the equation of the tangent line to the curve at the point on the curve where $\theta = \pi/3$.

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{\left(\sin\frac{\theta}{2}\right)\sin\theta + \left(1-2\cos\frac{\theta}{2}\right)\cos\theta}{\left(\sin\frac{\theta}{2}\right)\cos\theta - \left(1-2\cos\frac{\theta}{2}\right)\sin\theta}$$

$$\theta = \frac{\pi}{3}; \quad \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(1-\sqrt{3}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(1-\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right)} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{4}}{\frac{7}{4} - \frac{\sqrt{3}}{2}} = m \approx 0.076$$

Point: $\left(r, \frac{\pi}{3}\right) = \left(1 - 2\cos\frac{\pi}{6}, \frac{\pi}{3}\right) = \left(1-\sqrt{3}, \frac{\pi}{3}\right) \approx \left(-0.73, \frac{\pi}{3}\right)$

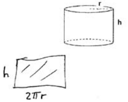
Cartesian: $(x, y) = \left(r\cos\frac{\pi}{3}, r\sin\frac{\pi}{3}\right) = \left((1-\sqrt{3})\frac{1}{2}, (1-\sqrt{3})\frac{\sqrt{3}}{2}\right) \approx (-0.4, -.6) = (c, d)$

equation of tangent: $y = d + m(x-c)$

$= mx + (d - mc)$

$\approx 0.076x - 0.57$

(6) [Marks: 10] A right circular cylinder is changing shape; both the radius and height are changing. If the radius is increasing at a rate of 2 cm/sec, how must the height change in order that the surface area of the cylinder (top, bottom, and side) is not changing when the radius is 4 cm and the height is 6 cm?



$$A = 2\pi r^2 + 2\pi h r$$

$$A' = 4\pi r r' + 2\pi h' r + 2\pi h r'$$

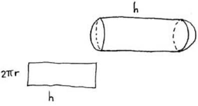
$$= 2\pi\left[(2r+h)r' + rh'\right]$$

$$A' = 0 \Rightarrow (2r+h)r' + rh' = 0$$

$$\rightarrow h' = -\frac{(2r+h)r'}{r}$$

$r' = 2,\ r = 4;\ h = 6$; $$h' = -\frac{(2(4)+6)\cdot 2}{4} = -7 \text{ cm/s}$$

(7) [Marks: 10] A fuel tank that will hold 100 cubic metres of fuel is being built in the shape of a cylindrical middle part capped by hemispheres at both ends. If material for the hemispherical ends costs twice as much as the material for the cylindrical middle part (dollars per square metre), find the shape (length and width) of the tank that minimizes the cost of material used to build it. The surface area of a sphere of radius r is $4\pi r^2$; its volume is $\frac{4}{3}\pi r^3$.



$$\text{vol} = V = \tfrac{4}{3}\pi r^3 + \pi r^2 h = 100$$

$$\Downarrow$$

$$\pi r h = \frac{100 - \frac{4}{3}\pi r^3}{r}$$

area $A = \underset{\text{cost}}{2}(4\pi r^2) + 2\pi r h$

$$A = 8\pi r^2 + \frac{200 - \frac{8}{3}\pi r^3}{r}$$

$$A' = 16\pi r - \frac{200}{r^2} - \frac{16}{3}\pi r$$

$$A' = 0;\ \left(16 - \frac{16}{3}\right)\pi = \frac{200}{r^3} \rightarrow r^3 = \frac{200}{\frac{32}{3}\pi};\ r \approx 1.8$$

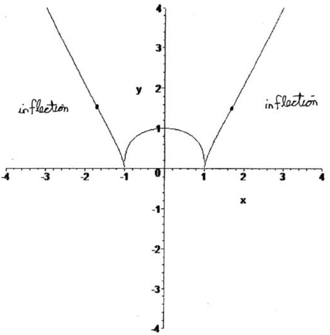
$$A'' = \frac{32}{3}\pi + \frac{400}{r^3} > 0 \Rightarrow \text{min.}$$

$$r = \sqrt[3]{\frac{200}{\frac{32}{3}\pi}} \approx 1.8,\quad h = \frac{100 - \frac{4}{3}\pi r^3}{\pi r^2} \approx 7.25$$

(8) [Marks: 10] Sketch the graph of the function $f(x) = (x^2 - 1)^{2/3}$. The first two derivatives are;

$$f'(x) = \frac{4}{3}\frac{x}{(x^2-1)^{1/3}}, \qquad f''(x) = \frac{4}{9}\frac{x^2-3}{(x^2-1)^{4/3}}$$

Be sure to locate intercepts, local extrema, and to indicate regions of concavity and points of inflection.



critical points ; $0, \pm 1$ first der test @ ± 1 $\Rightarrow$ min

second/first der test @ 0 $\Rightarrow$ max

inflection ; $f'' = 0$ at $x = \pm\sqrt{3}$; f'' changes sign ✓ $\Rightarrow$ inflection

Not at $x = \pm 1$

(9) [Marks: 6] Determine the slant asymptote(s) of the following function. Also determine how the graph approaches the asymptote. Illustrate your answer with a sketch.

$$y = (2x^3 - x^2 + 4)^{1/3} + 3$$

$$y' = \frac{1}{3}\left(2x^3 - x^2 + 4\right)^{-2/3}\left(6x^2 - 2x\right)$$

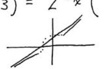
$$= \frac{3x^2\left(2x - \frac{2}{3x}\right)}{3x^2\left(2 - \frac{1}{x} + \frac{4}{x^2}\right)^{2/3}} = \frac{2x}{\left(2 - \frac{1}{x} + \frac{4}{x^2}\right)^{2/3}} - \frac{2/3x}{\left(2 - \frac{1}{x} + \frac{4}{x^2}\right)^{2/3}}$$

$$\rightarrow 2^{1/3}x \quad \text{as } x \rightarrow \pm\infty.$$

$$y - 2^{1/3}x = 2^{1/3}x\left(1 - \frac{1}{2x} + \frac{4}{x^3}\right)^{1/3} - 2^{1/3}x + 3 \longrightarrow 3 \text{ as } x \rightarrow \pm\infty$$

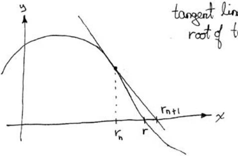
$\therefore$ slant asymptote is $2^{1/3}x + 3$.

$$y - \left(2^{1/3}x + 3\right) = 2^{1/3}x\left(\left[1 - \frac{1}{2x} + \frac{4}{x^3}\right]^{1/3} - 1\right) \longrightarrow \begin{cases} 0^- & x \rightarrow +\infty \\ 0^- & x \rightarrow -\infty \end{cases}$$



$y(x)$ below asymptote

(10) [Marks: 6] Make a sketch explaining Newton's method for finding roots of a function $f(x)$. Derive the formula used in Newton's method.

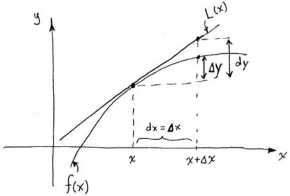


tangent line at $x = r_n$; $y = f(r_n) + f'(r_n)(x - r_n)$

root of tangent line; $0 = f(r_n) + f'(r_n)(r_{n+1} - r_n)$

$$\rightarrow \boxed{r_n = r_{n+1} - \frac{f(r_n)}{f'(r_n)}}$$

(11) [Marks: 12] (a) Make a sketch to illustrate what the linear approximation $L(x)$ (linearization) of a function $f(x)$ is at the point $x = a$. On your sketch indicate what the quantities $\Delta x, \Delta y, dx$ and dy are.



(b) Use differentials to estimate how accurate the measurement of the volume of a sphere must be so that the percentage error in the computed area of the sphere is no larger than 1%. (Take a look at Question 7 for some useful formulae.)

$$V = \frac{4}{3}\pi r^3 \quad ; \quad A = 4\pi r^2$$

$$A(V) = 4\pi\left(\frac{3}{4\pi}V\right)^{2/3}$$

$$dA = A' \, dV$$

$$= 4\pi\left(\frac{3}{4\pi}\right)^{2/3} \frac{2}{3} V^{-1/3} \, dV$$

$$\rightarrow \frac{dA}{A} = \frac{2}{3V} \, dV \le 0.01$$

$$\Rightarrow dV \le 0.015 V \qquad (15\% \text{ error in } V)$$

(12) [Marks: 8] On January 1 an amount of money is deposited into an account that compounds interest continuously. One half of a year later there is \$1,127.79 in the account, and one year after the deposit is made there is \$1144.83 in the account. How much will be in the account two years after the deposit is made?

$A(t) = A_0 e^{rt}$, r is interest rate per year

$A(\frac{1}{2}) = 1127.79 = A_0 e^{r/2} \rightarrow A_0 = 1127.79\, e^{-r/2}$

$A(1) = 1144.83 = A_0 e^{r} = 1127.79\, e^{r/2}$

$\Rightarrow \frac{r}{2} = \ln\left(\frac{1144.83}{1127.79}\right) \Rightarrow r = 0.03$

$\Rightarrow A_0 = \underline{1,111.00}$

(13) [Marks: 6] Identify the type of conic section whose equation is given here, and find the vertex (or vertices), foci, and directrix; $y^2 - 6x = 8y - 16$. Make a sketch of the curve.

$y^2 - 8y = 6x - 16$

$(y-4)^2 - 16 = 6x - 16$

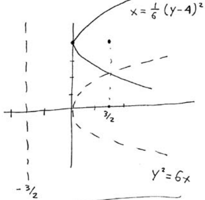
$x = \frac{1}{6}(y-4)^2$

This is curve of $x = \frac{1}{6}y^2$ shifted vertically by 4.

For $x = \frac{1}{6}y^2$; $y^2 = 6x = 4px$

$\rightarrow p = \frac{3}{2}$

directrix is $\underline{x = -\frac{3}{2}}$; focus $(\frac{3}{2}, 0)$



For $x = \frac{1}{6}(y-4)^2$;
directrix is $x = -\frac{3}{2}$
Focus $(\frac{3}{2}, 4)$
vertex $(0, 4)$