SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

Midterm 1

MATH 150 Fall 2006

Instructor: Dr. Mulholland

October 4, 2006, 8:30 – 9:20 a.m.

Name: Solutions (please print)

family name *given name*

SFU ID: ______________________

student number *SFU-email*

Signature: ______________________

Instructions:

1. Do not open this booklet until told to do so.
2. Write your name above in block letters. Write your SFU student number and email ID on the line provided for it.
3. Write your answer in the space provided below the question . If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
5. This exam has 5 questions on 8 pages (not including this cover page). Once the exam begins please check to make sure your exam is complete.
6. **No** calculators, books, papers, or electronic devices shall be within the reach of a student during the examination.
7. **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

Question	Maximum	Score
1	12	
2	8	
3	8	
4	6	
5	6	
Total	40	

1. Compute the following limits.

[3] (a) $\lim_{x\to -1}(x^3-2x^2+5x+1)^2$

$$= \left(\lim_{x\to -1} x^3-2x^2+5x+1\right)^2$$

$$= \left((-1)^3-2(-1)^2+5(-1)+1\right)^2$$

$$= (-1-2-5+1)^2$$

$$= 49$$

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[3] (b) $\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$

$$= \lim_{x\to 5}\left(\frac{\sqrt{x-1}-2}{x-5}\right)\left(\frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}\right)$$

$$= \lim_{x\to 5}\frac{(x-1)-4}{(x-5)(\sqrt{x-1}+2)}$$

$$= \lim_{x\to 5}\frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x-1}+2)}$$

$$= \lim_{x\to 5}\frac{1}{\sqrt{x-1}+2}$$

$$= \frac{1}{\sqrt{5-1}+2}$$

$$= 1/4.$$

[3] (c) $\lim_{x\to 7^-} \dfrac{x+1}{x-7}$

As $x \longrightarrow 7^-$, $x+1 \longrightarrow 8$ and $\underbrace{x-7 \longrightarrow 0^-}$

i.e. approaches 0 through negative numbers

$$\therefore \lim_{x\to 7^-} \frac{x+1}{x-7} = -\infty$$

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[3] (d) $\lim_{\theta\to 0} \dfrac{1+\theta^2-\cos^2\theta}{\theta^2}$

$$= \lim_{\theta\to 0} \frac{\theta^2+\sin^2\theta}{\theta^2}$$

$$= \lim_{\theta\to 0} \left[1 + \frac{\sin^2\theta}{\theta^2} \right]$$

$$= 1 + \Bigg[\underbrace{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}}_{=1}\Bigg]^2$$

$$= 1 + 1$$

$$= 2.$$

2. True or False. If True provide an explanation, if False give justification (for instance give an example for which the statement doesn't hold).

[2] (a) If f is one-to-one, then $f^{-1}(x) = \frac{1}{f(x)}$.

False.

Counter examples: (i) $f(x) = x$ has inverse $f^{-1}(x) = x$, NOT $\frac{1}{x}$

(ii) e^{x} has $\ln x$ as its inverse, NOT e^{-x}

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[2] (b) The function $g(x) = x\sin(x)$ is an even function.

True. The product of two odd functions (for example x & $\sin x$) is even.

In particular, $g(-x) = (-x)\sin(-x) = x\sin x = g(x)$

$\therefore$ g is even.

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[2] (c) Suppose that a continuous function $f(x)$ satisfies the following table of values.

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3
$f(x)$	-3	-1.25	0.25	0.75	1	0.75	-0.25	-1.25	-3

Then the function $f(x)$ has at least 2 zeros in the interval $(-1, 3)$.

True. By the Intermediate Value theorem there is a zero between -0.5 and 0 since $f(-0.5) < 0$ and $f(0) > 0$. Similarly, there is also a zero between 1.5 and 2.

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[2] (d) If f has domain $[0, \infty)$ and has no horizontal asymptote, then $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$.

False. Consider $f(x) = \sin x$.

[2] **3.** (a) State the definition of **continuity** for a function $f(x)$ at $x = a$.

$$\lim_{x \to a} f(x) = f(a)$$

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[3] (b) Suppose that

$$g(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 0, \\ bx^3 - 2 & \text{if } 0 < x \leq 1, \\ 2x - b & \text{if } x > 1, \end{cases}$$

where b is some constant.

Is the function g continuous at $x = 0$? Justify your answer.

$$\left.\begin{aligned} \lim_{x \to 0^-} g(x) &= \lim_{x \to 0^-} x^2 - 1 = -1 \\ \lim_{x \to 0^+} g(x) &= \lim_{x \to 0^+} bx^3 - 2 = -2 \end{aligned}\right\}$$

left and right-hand limits are <u>not</u> equal.

Therefore, $\lim_{x \to 0} g(x)$ does not exist.

$\therefore$ g is not continuous at $x = 0$.

[3] (c) Determine the constant b which makes the above function g continuous at $x = 1$.

The left-hand limit at 1 is :

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} bx^3 - 2 = b - 2$$

The right-hand limit at 1 is :

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 2x - b = 2 - b$$

g is continuous at $x = 1$ if and only if

$$b - 2 = 2 - b$$

$$\Rightarrow \quad 2b = 4$$

$$\Rightarrow \quad \boxed{b = 2}$$

[2] **4.** (a) State the **Squeeze Law**, clearly identifying any hypotheses and the conclusion.

If $f(x) \le g(x) \le h(x)$ for x near a (except possibly a itself) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then $$\lim_{x\to a} g(x) = L$$

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[4] (b) Use the Squeeze Law to compute $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)}$. Justify your answer.

Since

$$-1 \le \sin(\pi/x) \le 1$$

then

$$e^{-1} \le e^{\sin(\pi/x)} \le e$$ (since e^x an increasing function)

and so

$$\sqrt{x}\,e^{-1} \le \sqrt{x}\,e^{\sin(\pi/x)} \le \sqrt{x}\,e.$$

Taking limits we have

$$\lim_{x\to 0} \sqrt{x}\,e^{-1} = 0 \quad \& \quad \lim_{x\to 0} \sqrt{x}\,e = 0,$$

so by the Squeeze Law

$$\lim_{x\to 0} \sqrt{x}\,e^{\sin(\pi/x)} = 0$$

[4] **5.** (a) Find

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

where $f(x) = x^3 - 1$.

$$= \lim_{h\to 0} \frac{[(2+h)^3 - 1] - [2^3 - 1]}{h}$$

$$= \lim_{h\to 0} \frac{[\cancel{8} + 12h + 6h^2 + h^3 - \cancel{1}] - \cancel{7}}{h}$$

$$= \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h}$$

$$= \lim_{h\to 0} 12 + 6h + h^2$$

$$= 12.$$

[1] (b) What does the result in (a) tell you about the tangent line to the graph of $y = f(x)$ at $x = 2$?

The slope of the tangent line at $x = 2$ is 12.

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[1] (c) Find the equation of the tangent line to $y = f(x)$ at $x = 2$.

$$y - 7 = 12(x - 2)$$

$$y = 12x - 17$$