MATH 150
Midterm 2 Solutions, November 3, 2005

1) Find the indicated derivatives of the following functions. You do *not* need to simplify your answers.

(1a) (4 marks) $\quad y' \text{ and } y''$ where $y = e^{\sin(2x)}$

Answer

$$y' \;=\; 2\cos(2x)e^{\sin(2x)}$$

$$y'' \;=\; -4\sin(2x)e^{\sin(2x)} + 4\Big(\cos(2x)\Big)^2 e^{\sin(2x)}$$

(1b) (3 marks) $\quad g'(t)$ where $g(t) \;=\; \left(\frac{1}{t+1}\right)^{t^2}, \quad t > 0$

Solution:

Let $f(t) = \ln\Big(g(t)\Big) = t^2 \ln\left(\dfrac{1}{t+1}\right)$. Then $g'(t) = f'(t)g(t)$. We have,

$$f'(t) \;=\; 2t\ln\left(\frac{1}{t+1}\right) + t^2\Big(t+1\Big)\left(\frac{-1}{(t+1)^2}\right)$$

2) A particle travels along a straight line. The distance s (in metres) the particle is from the origin 0 at time t (in seconds) is given by

$$s(t) \;=\; t^4 - 4t^3 + 2$$

(2a) (1 mark) Find the velocity and acceleration at time t.

Answer

$$\begin{aligned} v(t) &= s'(t) = 4t^3 - 12t^2 = 4t^3(t-3) \\ a(t) &= s''(t) = 12t^2 - 24t = 12t(t-2) \end{aligned}$$

(2b) (3 marks) When is the particle speeding up? When is the particle slowing down?

Solution:

$v(t) > 0$ on $(3, \infty)$, and $v(t) < 0$ on $(-\infty, 0) \cup (0, 3)$.
$a(t) > 0$ on $(-\infty, 0)$, $(2, \infty)$, and $a(t) < 0$ on $(0, 2)$.

Particle is speeding up when $a(t) \cdot v(t) > 0$, and slowing down when $a(t) \cdot v(t) < 0$. Thus,

$$\begin{aligned} \text{speeding up} &: (0,2) \cup (3, \infty) \\ \text{slowing down} &: (-\infty, 0) \cup (2, 3) \end{aligned}$$

3) Consider the curve defined by $x^2 - y^2 = xy + 1$.

(3a) (1 mark) Show that $(1, -1)$ and $(-1, 1)$ lie on the curve.

Solution:

$$\begin{array}{ccrl} (1,-1) & : & \text{left side} & (1)^2 - (-1)^2 = 0 \\ & & \text{right side} & (1)(-1) + 1 = 0 \\ \\ (-1,1) & : & \text{left side} & (-1)^2 - (1)^2 = 0 \\ & & \text{right side} & (-1)(1) + 1 = 0 \end{array}$$

(3b) (1 mark) Find another point on the curve *different from* the points in (3a).

Solution:

Suppose $y = 0$. Then the equation defining the curve is $x^2 - 0 = 0 + 1 \longrightarrow x^2 = 1 \longrightarrow x = \pm 1$. So $(1, 0)$ and $(-1, 0)$ are on the curve. (In general, the point (a, b) is on the curve if $a^2 - b^2 = ab + 1$.)

(3c) (3 marks) Use implicit differentiation to find $y'$and y''. Express your answer in terms of x and y only. You do not need to simplify your answer.

Solution:

Differentiating the equation defining the curve with respect to x, and treating $y = y(x)$, we obtain,

$$2x - 2yy' = y + xy \quad \longrightarrow \quad y' = \frac{2x - y}{x + 2y}$$

Now, $y'' = \frac{d}{dx} y'$, so

$$\begin{aligned} y'' &= \frac{d}{dx}\left(\frac{2x - y}{x + 2y}\right) \\ &= \frac{(2 - y')(x + 2y) - (2x - y)(1 + 2y')}{(x + 2y)^2} \end{aligned}$$

now substitute for y' ;

$$= \frac{(2 - (\frac{2x-y}{x+2y}))(x + 2y) - (2x - y)(1 + 2(\frac{2x-y}{x+2y}))}{(x + 2y)^2}$$

(3d) (4 marks) Find all points on the curve whose tangent line has slope 1.

Solution:

Solve $y' = 1$;

$$\frac{2x - y}{x + 2y} = 1 \longrightarrow 2x - y = x + 2y \longrightarrow x = 3y$$

Now substitute $x = 3y$ into the equation defining the curve;

$$(3y)^2 - y^2 = (3y)y + 1 \longrightarrow 8y^2 = 3y^2 + 1 \longrightarrow 5y^2 = 1 \longrightarrow y = \pm\frac{1}{\sqrt{5}} \longrightarrow x = \pm\frac{3}{\sqrt{5}}$$

Thus, the points are $(\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, $(\frac{-3}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$

4) (6 marks) You are driving in a car at 60 km/hr towards the place where a rocket has recently been launched. You see the rocket directly in front of you an angle of elevation θ (see diagram). It is travelling at 300 km/hr vertically. When you are 10 km from the launch pad the rocket is at height 30 km. At what rate do you see the angle of elevation θ of the rocket changing at that time?

Solution:

Let $x(t)$ be the position of the car measured from the launch pad, and $y(t)$ be the height of the rocket. The relation between x, y, θ is

$$\tan\theta = \frac{y}{x}$$

We differentiate this to obtain relations between x', y', θ';

$$\frac{d}{dt}\Big(x\tan\theta = y\Big) \longrightarrow x'\tan\theta + x\theta'\sec^2\theta = y' \longrightarrow \theta' = \frac{y' - x'\tan\theta}{x\sec^2\theta}$$

Now substitute the values $x = 10, y = 30, x' = -60, y' = 300$ (so that $\tan\theta = 30/10$, $\sec^2\theta = 10$),

$$\theta' = \frac{(300) - (-60)(30/10)}{(10)(10)} = 1.2 \quad \text{radians per second}$$

5) (4 marks) Use the linear approximation of $f(x)$ at $x = 3$ to estimate $f(3.05)$ where

$$f(x) = x\sqrt{8x+1}.$$

Solution:

$$\begin{aligned}
f'(x) &= \sqrt{8x+1} + \frac{4x}{\sqrt{8x+1}} \\
f(3) &= 3\sqrt{25} = 15 \\
f'(3) &= \sqrt{25} + \frac{12}{\sqrt{25}} = 7.4 \\
L(x) &= f(3) + f'(3)(x-3) \\
L(3.05) &= 15 + 7.4(0.05) = 15.37
\end{aligned}$$

. $f(x) \approx L(x)$ for x near 3, so our estimate is $f(3.05) \approx L(3.05) = 15.37$.