MATH 150
Midterm 1, October 6, 2005 Solutions

1) (4 marks) Evaluate the following limit if it exists. If it does not exist, explain why. You must provide your reasoning which CANNOT be simply "this is what the calculator tells me."

$$\lim_{x\to\infty} x\left[\frac{1}{x}\right]$$

Here, $[x]$ is the greatest integer less than or equal to x (slightly different notation than the text). (So $\left[\frac{1}{x}\right]$ is the greatest integer less than or equal to $\frac{1}{x}$)

Solution:

If $x > 1$, $\frac{1}{x} < 1$, so $\left[\frac{1}{x}\right] = 0$. Thus, if $x > 1$, $\;x\left[\frac{1}{x}\right] = x \cdot 0 = 0$, and so

$$\lim_{x\to\infty} x\left[\frac{1}{x}\right] \;=\; 0$$

2a) (2 marks) State the Intermediate Value Theorem.

Solution:

If $f(x)$ is continuous on $[a,b]$ with $f(a) \neq f(b)$ and if N is any number between $f(a)$ and $f(b)$, then there is a $c \in (a,b)$ such that $f(c) = N$.

2b) (4 marks) Use the Intermediate Value Theorem to find an interval of length less than or equal to $\frac{1}{4}$ that contains a root of the equation

$$f(x) = 2x^3 - 3x + 4$$

Explain your answer completely.

Solution:

First, $f(x)$ is a continuous function (so it will be continuous on any interval $[a,b]$). Here, $N = 0$. So to use the IVT, we need to find a and b such that $f(a) \cdot f(b) < 0$ (then N is between $f(a)$ and $f(b)$).
$f(0) > 0$ and $f(-2) < 0$, so by IVT there is a root of $f(x)$ in $(-2, 0)$. This interval is of length 2, though. So we need to refine our estimate (find a smaller interval). The easiest thing to do is to divide our initial interval $(-2, 0)$ in half. So we look at $x = -1$.
$f(-1) > 0$, so by the IVT there is a root in $(-2, -1)$. This interval is of length 1.
$f(-\frac{3}{2}) > 0$, so by IVT there is a root in $(-2, -\frac{3}{2})$. This interval is of length $\frac{1}{2}$.
$f(-\frac{7}{4}) < 0$, so by IVT there is a root in $(-\frac{7}{4}, -\frac{3}{2})$. This interval is of length $\frac{1}{4}$.

Our estimate for the root is then $x = -13/8$ (the midpoint of $(-\frac{7}{4}, -\frac{3}{2})$) with an error of no more than $\pm\frac{1}{8}$.

3) (4 marks) Find the equation of the tangent line to the graph of $f(x) = 2x^2 - \sqrt{x} + 1$ at the point $(1, 2)$.

Solution:

$f'(x) = 4x - \dfrac{1}{2\sqrt{x}}$. So $f'(1) = 4 - \frac{1}{2} = \frac{7}{2}$. This is the slope of the tangent line.

Using the point-slope formula, with $m = \frac{7}{2}$ and the point $(x_1, y_1) = (1, 2)$;

$$\frac{7}{2} = \frac{y-2}{x-1} \implies y = \frac{7}{2}x - \frac{3}{2}$$

4) (4 marks) Below is the graph of a function $f(x)$. Sketch the graph of the <u>derivative</u> function $f'(x)$ in the space below it (use the axis that are drawn).

Solution:

See course web page (in the midterms folder) for this image.

5) (5 marks) Find all vertical and horizontal asymptotes (if there are any) of the following function;

$$g(x) = \frac{2\sqrt[4]{x^4} - 5x + 1}{x+1}$$

Explain your answers completely.

Solution:

There is a vertical asymptote at $x = -1$ because the denominator is 0 there and the numerator is not 0 there (it is 6). In this case $\lim\limits_{x \to -1^{\pm}} g(x) = \pm\infty$ (check!).

For horizontal asymptotes;

$$\begin{aligned}
\lim_{x \to -\infty} g(x) &= \lim_{x \to -\infty} \frac{2(-x) - 5x + 1}{x+1}, \qquad \text{because } \sqrt[4]{x^4} = -x \text{ if } x < 0 \\
&= \lim_{x \to -\infty} \frac{x(-2 - 5 + \frac{1}{x})}{x(1 + \frac{1}{x})} \\
&= \lim_{x \to -\infty} \frac{-7 + \frac{1}{x}}{1 + \frac{1}{x}} \\
&= -7
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to \infty} g(x) &= \lim_{x \to -\infty} \frac{2(x) - 5x + 1}{x+1}, \qquad \text{because } \sqrt[4]{x^4} = x \text{ if } x > 0 \\
&= \lim_{x \to \infty} \frac{x(2 - 5 + \frac{1}{x})}{x(1 + \frac{1}{x})} \\
&= \lim_{x \to \infty} \frac{-3 + \frac{1}{x}}{1 + \frac{1}{x}} \\
&= -3
\end{aligned}$$

Thus, $g(x)$ has horizontal asymptote $y = -7$ as $x \to -\infty$, and horizontal asymptote $y = -3$ as $x \to \infty$.

6a) (2 marks) Below is the graph of a function $f(x)$ where $\lim_{x\to 2} f(x) = 3$. An ε_1 is given and the interval $|f(x) - 3| < \varepsilon_1$ is indicated.

On the x-axis, indicate an interval $|x - 2| < \delta_1$, for some δ_1, such that if $|x - 2| < \delta_1$, then $|f(x) - 3| < \varepsilon_1$. Explain why your δ_1 works.

Solution:

See image posted on the course web page.

6b) (5 marks) Using the precise $\varepsilon - \delta$ definition of a limit, prove that

$$\lim_{x\to 2}(2x^2 - 3x + 1) \;=\; 3.$$

Solution:

Recall the precise definition of a limit: $\lim_{x\to a} f(x) = L$ means that for all $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

So let $\varepsilon > 0$.

$$\begin{aligned} |f(x) - L| &= |2x^2 - 3x - 2| \\ &= 2|x-2||x + \frac{1}{2}|. \end{aligned}$$

Now we need to estimate $|x + \frac{1}{2}|$. So suppose that $|x - 2| < 1$ (so that $x \in (1,3)$). Then for these x'^s, $|x + \frac{1}{2}| < \frac{7}{2}$. So in this case

$$|f(x) - L| \;=\; 2|x-2||x + \frac{1}{2}| \;< |x-2| \cdot \frac{7}{2}$$

Thus, if $|x - 2| < 1$ <u>and</u> $\delta = \frac{2\varepsilon}{7}$, then

$$|f(x) - L| \;=\; 2|x-2||x+\frac{1}{2}| \;\;<\;\; |x-2| \cdot \frac{7}{2} \;<\; \delta \cdot \frac{7}{2} \;=\; \frac{2\varepsilon}{7} \cdot \frac{7}{2} \;=\; \varepsilon$$

i.e., $|f(x) - L| = |2x^2 - 3x - 2| < \varepsilon$.

Now, the above answer for δ is only valid if $\varepsilon < \frac{7}{2}$ because if $\varepsilon \geq \frac{7}{2}$, $\delta = \frac{2\varepsilon}{7} \geq 1$ and so $|x - 2|$ may <u>not</u> be < 1 for all the x'^s for which $|x - 2| < \delta$, which we used to obtain the answer $\delta = \frac{2\varepsilon}{7}$. So for $\varepsilon \geq \frac{7}{2}$ we just choose $\delta = 1$ ($\delta = 1$ works when $\varepsilon = \frac{7}{2}$ so it works for any larger ε).

So our final answer is $\delta = \min\left(1,\; \dfrac{2\varepsilon}{7}\right)$.