SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Final Exam

MATH 150 Fall 2007

Instructor: **(CIRCLE ONE)** Dr. Mulholland & Dr. Goddyn

December 13, 2007, 7:00 – 10:00 p.m.

Name: ______________________________ (please print)
family name *given name*

SFU ID: ______________________________
student number *SFU-email*

Signature: ______________________________

Instructions:

1. Do not open this booklet until told to do so.
2. Write your name above in block letters. Write your SFU student number and email ID on the line provided for it.
3. Write your answer in the space provided below the question. If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
5. This exam has 11 questions on 13 pages (not including this cover page). Once the exam begins please check to make sure your exam is complete.
6. **No** calculators, books, papers, or electronic devices shall be within the reach of a student during the examination. Leave answers in "calculator ready" expressions: such as $3 + \ln 7$ or $e^{\sqrt{2}}$.
7. **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

Question	Maximum	Score
1	12	
2	9	
3	9	
4	9	
5	9	
6	9	
7	9	
8	10	
9	10	
10	10	
11	0	
Total	96	

[2] **1.** (a) State the **Intermediate Value Theorem**, clearly identifying any hypotheses and the conclusion.

If f is continuous on the closed interval $[a,b]$ and N is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there exists a number c in (a,b) such that $f(c) = N$.

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[2] (b) State the definition of **derivative** of a function f at a number a.

The derivative of a function f at a number a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

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[2] (c) State the definition of a **critical number** of a function f.

A critical number is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

[2] (d) State the **Mean Value Theorem**, clearly identifying any hypotheses and the conclusion.

If f is a function such that

(i) f is continuous on the closed interval $[a,b]$

(ii) f is differentiable on the open interval (a,b)

then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

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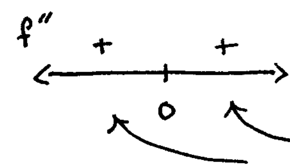
[2] (e) Give an example of a function f and a value $x = a$ such that $f''(a) = 0$ but f does *not* have an inflection point at $x = a$.

$$f(x) = x^4$$

$f''(x) = 12x^2$ so



concave up on both intervals $(-\infty, 0)$ and $(0,\infty)$

hence f does not have an inflection point at $x=0$.

Note $f''(0) = 0$.

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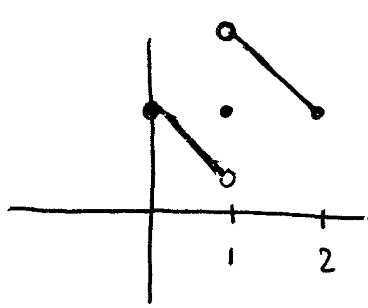
[2] (f) Give an example of a function defined everywhere on the closed interval $[0,2]$ but with no absolute maximum on that interval.



No max or min

but is defined everywhere.

Note: Any example of this must be discontinuous by the Extreme Value theorem.

Questions 2 - 7 are **Short-Answer Questions**. Each part of each question is worth 3 marks, but not all questions are of equal difficulty. Unless otherwise stated, it is not necessary to simplify your answers for these question. To receive full credit you must justify your answers.

[3] 2. (a) Evaluate $\lim_{x\to-\infty} \frac{(1-x^2)(2-x^2)}{5+x-3x^4}$.

$$\lim_{x\to-\infty} \frac{(1-x^2)(2-x^2)}{5+x-3x^4} = \frac{(-1)(-1)}{3} = \frac{1}{3}$$

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[3] (b) Evaluate $\lim_{x\to\pi/2^-} \left(x-\frac{\pi}{2}\right)\tan x$.

$$\lim_{x\to\pi/2^-} \frac{(x-\pi/2)\sin x}{\cos x} \begin{matrix}\to 0\\ \to 0\end{matrix} \overset{\text{by L'H}}{=} \lim_{x\to\pi/2^-} \frac{\sin x + (x-\pi/2)\cos x}{-\sin x}$$

$$= \frac{1+0}{-1} = -1.$$

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[3] (c) Evaluate $\lim_{x\to\infty} x^3 e^{-x+5}$.

$$\lim_{x\to\infty} \frac{x^3}{e^{x-5}} \begin{matrix}\to\infty\\ \to\infty\end{matrix} \overset{\text{by L'H}}{=} \lim_{x\to\infty} \frac{3x^2}{e^{x-5}} \to \frac{\infty}{\infty}$$ so apply L'H again

$$\overset{\text{by L'H}}{=} \lim_{x\to\infty} \frac{6x}{e^{x-5}} \to \frac{\infty}{\infty}$$ so apply L'H again

$$\overset{\text{by L'H}}{=} \lim_{x\to\infty} \frac{6}{e^{x-5}}$$

$$= 0.$$

[3] **3.** (a) Find the equation of the tangent line to the curve $y = (2+x)e^{-x}$ at the point (0,2)

$$y' = e^{-x} + (2+x)e^{-x}\cdot(-1) = -e^{-x}(1+x)$$

slope of tangent at (0,2) : $y'|_{x=0} = -e^{-0}(1+0) = -1$

tangent line : $y-2 = (-1)(x-0)$

$$\boxed{y = -x+2}$$

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[3] (b) Find an equation of the slant asymptote to the curve $y = \dfrac{2x^3+3x^2+3x+6}{x^2+x+1}$.

$$\begin{array}{r|l} & 2x+1 \\ x^2+x+1 & 2x^3+3x^2+3x+6 \\ & 2x^3+2x^2+2x \\ \hline & x^2+x+6 \\ & x^2+x+1 \\ \hline & 5 \end{array}$$

Slant asymptote :

$$\boxed{y = 2x+1}$$

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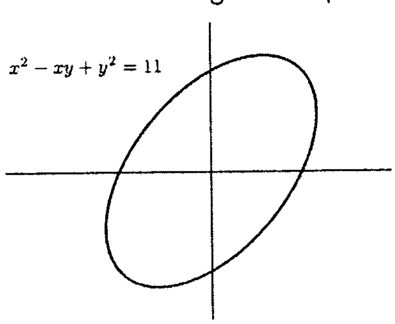
[3] (c) Find the point on the curve $x^2 - xy + y^2 = 11$ which lies in the first quadrant and has tangent line parallel to the line $y = -x-2$.



$$x^2 - xy + y^2 = 11$$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

Want $\frac{dy}{dx} = -1 \Rightarrow \frac{y-2x}{2y-x} = -1 \Rightarrow y-2x = -2y+x$

$\Rightarrow 3y = 3x \Rightarrow y = x$

Plugging $x=y$ into the original equation of the curve we find $x^2 - x\cdot x + x^2 = 11 \Rightarrow x = \sqrt{11}$ & $y = \sqrt{11}$.

[3] **4.** (a) If $y = (\cos x)^x$, find the derivative dy/dx.

$$\ln y = x \ln(\cos x)$$

Taking $\frac{d}{dx}$:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\cos x) + x \cdot \frac{1}{\cos x} \cdot (-\sin x)$$

$$\frac{dy}{dx} = y \left(\ln(\cos x) - \frac{x \sin x}{\cos x} \right)$$

$$\frac{dy}{dx} = (\cos x)^x \left(\ln(\cos x) - \frac{x \sin x}{\cos x} \right)$$

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[3] (b) Show that the equation $x^4 = x + 1$ has **exactly** one solution in the interval $[1, 2]$.

Let $f(x) = x^4 - x - 1$, then it suffices to show $f(x) = 0$ has exactly one solution.

Since $f(1) = -1$, $f(2) = 13$ then the Intermediate Value Theorem implies there is at least one solution.

Moreover, $f'(x) = 4x^3 - 1$ is positive on the interval $[1,2]$ hence f is strictly increasing on this interval and thus can only cross the axis at most once.

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[3] (c) Use Newton's Method with the initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^4 = x + 1$.

As in part (b) we let $f(x) = x^4 - x - 1$.

Newtons iterative formula for successive approximations is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^4 - x_n - 1}{4x_n^3 - 1}$$

Therefore,

$$x_1 = 1$$

$$x_2 = 1 - \frac{1 - 1 - 1}{4 - 1} = 1 - \frac{-1}{3} = \frac{4}{3}.$$

[3] 5. (a) Let $f(x) = x^3 \tan^2(5x)$. Find $f'(x)$.

$$f'(x) = 3x^2 \tan^2(5x) + x^3 \frac{d}{dx}\left(\tan^2(5x)\right)$$
$$= 3x^2 \tan^2(5x) + x^3 \left(2\tan(5x) \cdot \frac{d}{dx}\tan(5x)\right)$$
$$= 3x^2 \tan^2(5x) + x^3 \left(2\tan(5x) \cdot \sec^2(5x) \cdot 5\right)$$
$$= 3x^2 \tan^2(5x) + 10x^3 \tan(5x)\sec^2(5x).$$

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[3] (b) Suppose f is a function satisfying $f(5) = 2$ and $f'(5) = 4$. Using a linear approximation to f at $x = 5$, find an approximation to $f(4.9)$.

$$f(4.9) \approx f(5) + f'(5)(4.9 - 5)$$
$$= 2 + 4(-0.1)$$
$$= 2 - 0.4$$
$$= 1.6.$$

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[3] (c) Find a function f such that the curve $y = f(x)$ satisfies $\frac{d^2y}{dx^2} = 12x$, passes through the point $(0, 1)$, and has a horizontal tangent there.

$$\frac{d^2y}{dx^2} = 12x$$

$\Rightarrow \quad \frac{dy}{dx} = 6x^2 + a \quad \rightsquigarrow$ Want $\left.\frac{dy}{dx}\right|_{x=0} = 0$, hence $a=0$

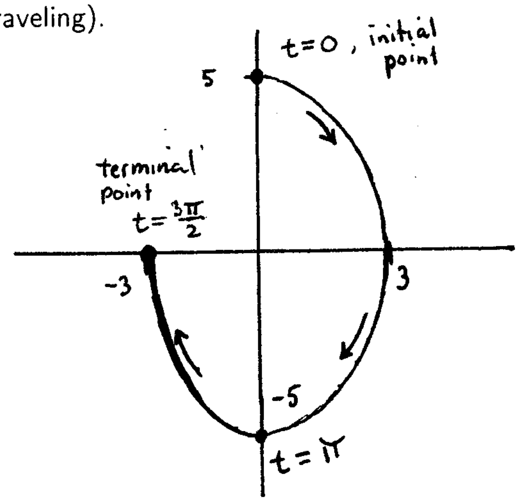
$\Rightarrow \quad y = 2x^3 + ax + b \quad \rightsquigarrow$ want $\left.y\right|_{x=0} = 1$, hence $b=1$

Therefore, $y = 2x^3 + 1$ or $\boxed{f(x) = 2x^3 + 1}$

[3] **6.** (a) Sketch the curve which is given by the parametric equations

$$x = 3\sin(t), \quad y = 5\cos(t), \quad 0 \leq t \leq 3\pi/2.$$

Clearly label the initial and terminal points and describe the motion of the point $(x(t), y(t))$ as t varies in the given interval (i.e. use arrows to indicate the direction the point is traveling).



Curve lies on the ellipse

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = 1 .$$

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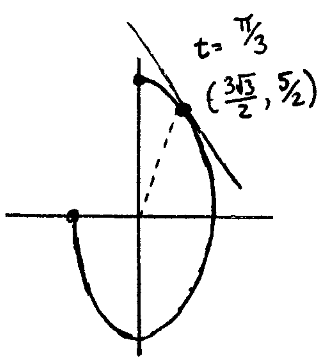
[3] (b) Find the slope of the tangent line to the curve in part (a) at the point $(x, y) = (\frac{3\sqrt{3}}{2}, \frac{5}{2})$.

The point $\left(\frac{3\sqrt{3}}{2}, \frac{5}{2}\right)$ corresponds to the value of the parameter $t = \pi/3$.



Slope of the tangent line to a parametric curve is :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-5\sin t}{3\cos t}$$

so,

$$\left.\frac{dy}{dx}\right|_{t=\pi/3} = \frac{-5\sin(\pi/3)}{3\cos(\pi/3)} = \frac{-5\left(\sqrt{3}/2\right)}{3\left(1/2\right)}$$

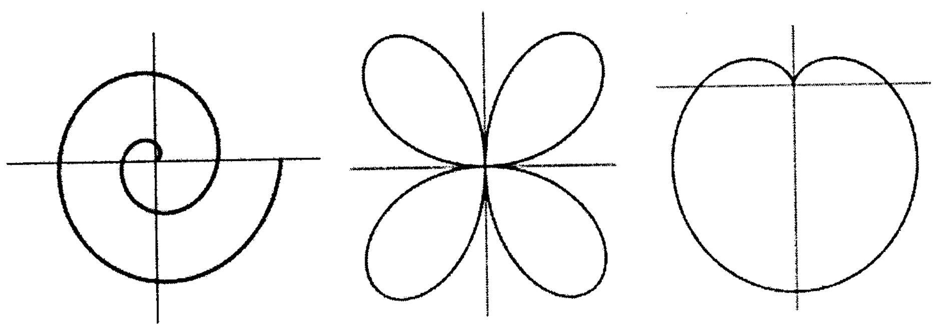
$$= -5/\sqrt{3} .$$

[3] (c) The graphs of the equations

(i) $r = 3\sin(2\theta)$

(ii) $r = 1 \quad \sin(\theta)$

are drawn below in polar coordinates. An extra graph is drawn as well. Match the equation with its corresponding graph and write your choice (either **(i)**, **(ii)** or **neither**) in the box under the graph.



neither	(i)	(ii)

7. A bank account initially contains \$1,000 and has an annual interest rate of 5%. Assume no additional deposits or withdrawals are made.

[3] (a) If the interest is compounded **annually**, write a formula for the balance at the end of t years.

Let $A(t)$ be the amount after time t, then

$$A(t) = 1000(1+r)^t$$
$$= 1000(1+0.05)^t$$

[3] (b) If the interest is compounded **semi-annually**, write a formula for the balance at the end of t years.

$$A(t) = 1000\left(1+\frac{0.05}{2}\right)^{2t}$$

[3] (c) If the interest is compounded **continuously**, write a formula for the balance at the end of t years.

$$A(t) = 1000\,e^{-0.05t}$$

Long-Answer Problems: In questions 8-11, to receive full credit you must justify your answers and show all your work.

8. Consider the function $f(x) = x^3 e^{-x+5}$.

[5] (a) Find and classify the critical points of f as either *local maximum*, *local minimum*, or *neither*.

$$f'(x) = 3x^2 e^{-x+5} + x^3 e^{-x+5} \cdot (-1) = e^{-x+5}(3x^2 - x^3)$$
$$= e^{-x+5} x^2 (3 - x)$$

Critical points : $x = 0, 3$

f': $+$ $\quad 0 \quad$ $+$ $\quad 3 \quad$ $-$

f has <u>neither</u> a local max or min at $x = 0$.

f has a <u>local max</u> at $x = 3$.

[5] (b) f has three inflection points, find them. Indicate the intervals where the graph of the function is concave-up and concave-down.

$$f''(x) = -e^{-x+5}(3x^2 - x^3) + e^{-x+5}(6x - 3x^2)$$
$$= e^{-x+5}(-3x^2 + x^3 + 6x - 3x^2)$$
$$= e^{-x+5}(x^3 - 6x^2 + 6x)$$
$$= e^{-x+5}\, x\,(x^2 - 6x + 6)$$

$f''(x) = 0 \Rightarrow x = 0$ or $x^2 - 6x + 6 = 0 \Rightarrow x = \frac{6 \pm \sqrt{36 - 4 \cdot 6}}{2} = \frac{6 \pm 2\sqrt{3}}{2}$

$x = 3 \pm \sqrt{3}$

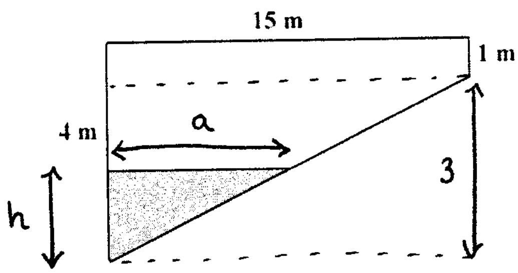
f'': $-$ $\quad 0 \quad$ $+$ $\quad 3-\sqrt{3} \quad$ $-$ $\quad 3+\sqrt{3} \quad$ $+$

Concave up on $(0, 3-\sqrt{3})$, $(3+\sqrt{3}, \infty)$

Concave down on $(-\infty, 0)$, $(3-\sqrt{3}, 3+\sqrt{3})$

Inflection points : $0,\ 3-\sqrt{3},\ 3+\sqrt{3}$

[10] 9. A swimming pool is 15 m long and 5 m wide Its depth varies uniformly from 1 m at the shallow end to 4 m at the deep end. Suppose that the pool is being filled at the rate of 10 m^3/min. At what rate is the depth of the water at the deep end increasing when the depth there is 2 m? (The figure shows a cross section of the pool.)



Let V be the volume of water at time t. Let h be the height of the water at time t.

Relationship between V and h:

$$V = [\text{Area of triangle}] \cdot [\text{width of pool}] = \frac{1}{2} ha \cdot 5$$

By similar triangles $\frac{a}{h} = \frac{15}{3} \Rightarrow a = 5h$

So,

$$V = \frac{25}{2} h^2$$

Differentiating with respect to time,

$$\frac{dV}{dt} = \frac{25}{2} \cdot 2h \cdot \frac{dh}{dt}$$

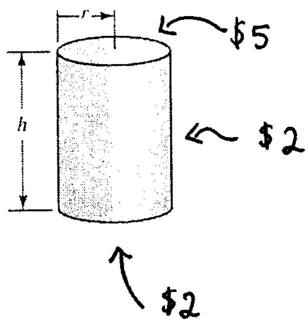
$$\frac{dV}{dt} = 25h \cdot \frac{dh}{dt}$$

Substitute in data: $\frac{dV}{dt} = 10$, $h = 2$

$$\left.\frac{dh}{dt}\right|_{h=2} = \frac{1}{25h} \cdot \frac{dV}{dt} = \frac{1}{25 \cdot 2} \cdot 10$$

$$\boxed{\left.\frac{dh}{dt}\right|_{h=2} = \frac{1}{5} \text{ m/min}}$$

[10] **10.** A storage container is to be made in the form of a right circular cylinder and have a volume of 28π m^3. Material for the top of the container costs \$5 per square metre and material for the side and base costs \$2 per square metre. What dimensions will minimize the total cost of the container?



Let C be the cost of the container.

$$C = 5\left[\text{Area of top}\right] + 2\left[\text{Area of side}\right] + 2\left[\text{Area of base}\right]$$
$$= 5\pi r^2 + 2(2\pi r \cdot h) + 2(\pi r^2)$$
$$= 7\pi r^2 + 4\pi r h$$

Volume of container:

$$V = 28\pi$$
$$\pi r^2 h = 28\pi \quad \Longrightarrow \quad h = \frac{28}{r^2}$$

Therefore,

$$C = 7\pi r^2 + \frac{4\cdot 28\pi}{r}$$

We want to minimize C on the interval $(0, \infty)$.

$$C'(r) = 14\pi r - \frac{4\cdot 28\pi}{r^2} = 14\pi\,\frac{(r^3 - 8)}{r^2}$$

Critical point: $r = 2$.

Since

$$\left.\begin{array}{l} \lim_{r\to 0} C(r) = +\infty \\ C(2) = 28\pi + 56\pi \\ \lim_{r\to\infty} C(r) = +\infty \end{array}\right\}$$

Therefore, C is minimum when $r = 2$, and $h = 7$.

[0] 11. **BONUS [4 points]** A number a is called a **fixed point** of a function f if $f(a) = a$. Show that if $f'(x) = 1$ for all real numbers x, then f has at most one fixed point.

Suppose a and b are fixed points, then by the mean value theorem there is a c in [a,b] such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{b-a}{b-a} \quad \text{, since } a, b \text{ are fixed points}$$

$$= 1.$$

But this contradicts the hypothesis. Therefore, there is at most one fixed point.