SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Final Exam

MATH 150 Fall 2006

Instructor: Dr. Mulholland

December 14, 2006, 3:30 – 6:30 p.m.

Name: Solutions (please print)

family name *given name*

SFU ID:

student number *SFU-email*

Signature:

Instructions:

1. Do not open this booklet until told to do so.
2. Write your name above in block letters. Write your SFU student number and email ID on the line provided for it.
3. Write your answer in the space provided below the question . If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
5. This exam has 12 questions on 13 pages (not including this cover page). Once the exam begins please check to make sure your exam is complete.
6. **No** calculators, books, papers, or electronic devices shall be within the reach of a student during the examination. Leave answers in "calculator ready" expressions: such as $3 + \ln 7$ or $e^{\sqrt{2}}$.
7. **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

Question	Maximum	Score
1	12	
2	9	
3	9	
4	9	
5	9	
6	6	
7	10	
8	6	
9	10	
10	10	
11	6	
12	4	
Total	100	

[2] **1.** (a) State the definition of the **limit** of $f(x)$ as x approaches a.

We write $\lim_{x \to a} f(x) = L$ and say the limit of f as x approaches a is L provided we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a.

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[2] (b) State the definition of **derivative** of a function f at a number a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists

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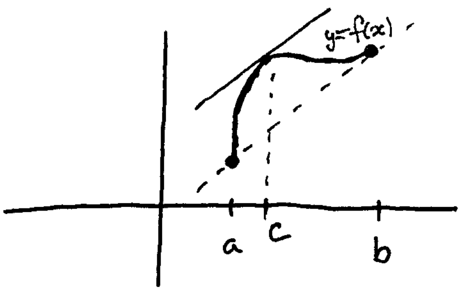
[2] (c) State the definition of a **critical number** of a function f.

A number x, in the domain of f, is a critical number of f if either

$$f'(x) = 0$$

or $f'(x)$ does not exist.

[2] (d) State the **Mean Value Theorem**, clearly identifying any hypotheses and the conclusion.



If f is continuous on $[a,b]$ and differentiable on (a,b) then there exists c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

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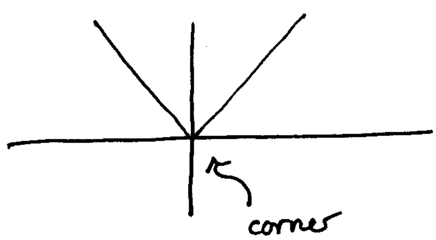
[2] (e) Give an example of a function that is continuous but not differentiable at a point.

$$f(x) = |x|$$



continuous at $x=0$

not differentiable at $x=0$

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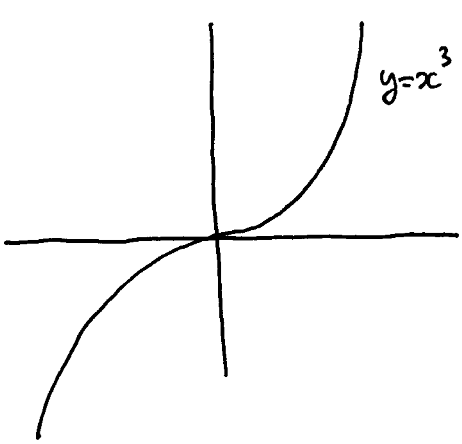
[2] (f) Give an example of a function with a critical number but no maximum or minimum.

$f(x) = x^3$ on $\mathbb{R}$



critical number is $x=0$

but no maximum or minimum.

Questions 2 - 6 are **Short-Answer Questions**. Each question is worth 3 marks, but not all questions are of equal difficulty. Unless otherwise stated, it is not necessary to simplify your answers for these question.

[3] **2.** (a) Evaluate $\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)}$$

$$= \lim_{x\to 1} \frac{x+1}{x-2}$$

$$= \frac{2}{-1} = \boxed{-2}$$

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[3] (b) Evaluate $\lim_{x\to \infty} \frac{x^2-1}{x^2-3x+2}$

$$= \frac{1}{1} = \boxed{1}$$

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[3] (c) Calculate the derivative of $f(x) = \sin^{-1}(x^2)$. [*Note*: Another notation for $\sin^{-1}$ is arcsin.]

$$f'(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2)$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

[3] **3.** (a) The curve $y = xe^{-2x}$ has one inflection point. Find the x coordinate of this point.

$$y' = e^{-2x} + xe^{-2x}(-2)$$
$$= e^{-2x} - 2xe^{-2x}$$

~~critical~~
inflection point : $y''=0$

$$y'' = -2e^{-2x} - 2e^{-2x} + -2xe^{-2x}(-2)$$
$$= -4e^{-2x} + 4xe^{-2x}$$
$$= -4e^{-2x}(1-x)$$

$$\boxed{x = 1}$$

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[3] (b) Find an equation of the slant asymptote to the curve $y = \dfrac{x^3+x^2}{x^2+1}$.

$$\begin{array}{r} x+1 \\ x^2+1 \,\overline{)\, x^3 + x^2} \\ \underline{x^3 + x} \\ x^2 - x \\ \underline{x^2 + 1} \\ -x-1 \end{array}$$

$$\therefore \quad \frac{x^3+x^2}{x^2+1} = x+1 - \frac{x+1}{x^2+1}$$

$\Rightarrow$ slant asymptote

$$\boxed{y = x+1}$$

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[3] (c) Find an equation of the tangent line at $(0,1)$ to the curve $x^3 + y^3 = \sin x + 1$.

$$3x^2 + 3y^2 \cdot y' = \cos x$$

$$y' = \frac{\cos x - 3x^2}{3y^2}$$

@ $(0,1)$ $\quad y' = \dfrac{\cos(0) - 3(0)}{3} = 1/3$

tangent line :

$$y - 1 = \frac{1}{3}(x - 0)$$

$$\boxed{y = \frac{1}{3}x + 1}$$

[3] **4.** (a) Find a number x_0 between 0 and $\pi/2$ such that the tangent line to the curve $y = \cos x$ at $x = x_0$ is parallel to the line $y = -x/2$.

$$y' = -\sin x$$

Want to find x such that $-\sin x = -\frac{1}{2}$

$$\Rightarrow \sin x = \frac{1}{2}$$

$$\Rightarrow \boxed{x = \frac{\pi}{6}}$$

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[3] (b) If $y = x^{\sin x}$, find the derivative dy/dx.

logarithmic differentiation :

$$\ln y = \sin x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \cdot \ln x)$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$\boxed{y' = x^{\sin x}\left(\cos x \cdot \ln x + \frac{1}{x}\sin x\right)}$$

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[3] (c) Using linear approximation, estimate $f(4.1)$, given that $f(4) = 3$ and $f'(x) = \sqrt{x^2 + 9}$.

linear approximation at $x = a$:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Here, $a = 4$ and $f'(4) = \sqrt{4^2 + 9} = 5$, thus

$$f(4.1) \approx f(4) + f'(4)(4.1 - 4)$$

$$= 3 + 5(0.1)$$

$$= 3.5$$

[3] 5. (a) Evaluate $\lim_{x \to 0} \frac{e^x - 1}{\sin(2x)}$. $0/0$ use L'Hospitals rule

$$\overset{\text{by L'h}}{=} \lim_{x \to 0} \frac{e^x}{2\cos(2x)}$$

$$= \frac{1}{2}$$

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[3] (b) If $f'(x) = 6x^2 + \sin x$ and $f(0) = 1$ find f.

$f(x) = 2x^3 - \cos x + C$ by antidifferentiation

We want $f(0) = 1$ so,

$1 = f(0) = 2(0)^3 - \cos(0) + C$

$\Rightarrow 1 = -1 + C$

$\Rightarrow C = 2$ $\therefore f(x) = 2x^3 - \cos x + 2$

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[3] (c) The function f is defined by

$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ Ax + B & \text{if } x \geq 1 \end{cases}$$

where A and B are constants. For what values of A and B is $f(x)$ differentiable for all x?

f is certainly continuous and differentiable everywhere except possibly at $x = 1$. To be differentiable it first must be continuous and f is continuous at $x = 1$ when

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^-} x^3 = \lim_{x \to 1^+} Ax + B$$

$$\boxed{1 = A + B} \longrightarrow \therefore A = 3,\ B = -2$$

To be differentiable at $x = 1$ we need

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$$

$$\lim_{x \to 1^-} 3x^2 = \lim_{x \to 1^+} A \Rightarrow \boxed{A = 3}$$

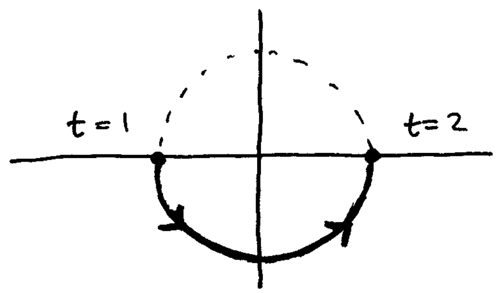
[3] **6.** (a) Sketch the curve which is given by the parametric equations

$$x = \cos(\pi t), \quad y = \sin(\pi t), \quad 1 \leq t \leq 2.$$

Clearly label the initial and terminal points and describe the motion of the point $(x(t), y(t))$ as t varies in the given interval (i.e. use arrows to indicate the direction the point is traveling).

$$x^2 + y^2 = \cos^2(\pi t) + \sin^2(\pi t) = 1$$

$\therefore$ curve lies on the circle of radius 1 centered at the origin



initial point:

$x(1) = \cos(\pi \cdot 1) = -1$

$y(1) = \sin(\pi \cdot 1) = 0$

terminal point:

$x(2) = \cos(2\pi) = 1$

$y(2) = \sin(2\pi) = 0$

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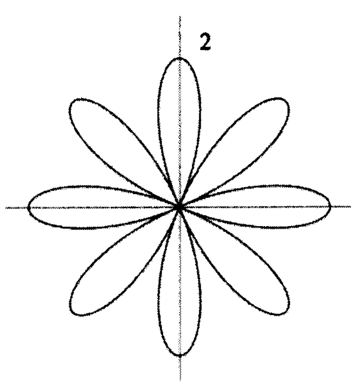
[3] (b) The graphs of the equations

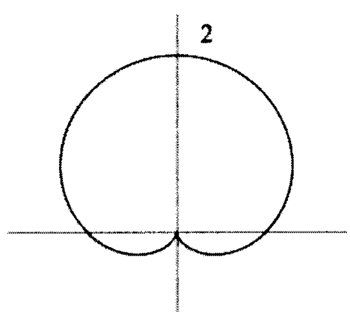
(i) $r = 1 + \sin\theta$

(ii) $r = 2\cos(4\theta)$

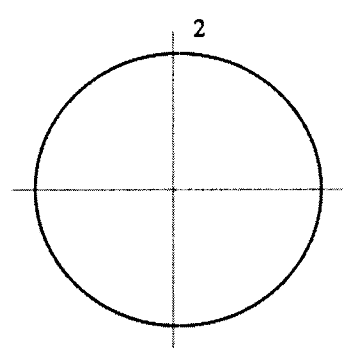
are drawn below in polar coordinates. An extra graph is drawn as well. Match the equation with its corresponding graph and write your choice (either **(i)**, **(ii)** or **neither**) in the box under the graph.



ii



i



neither

Full-Solution Problems: In questions 7-12, justify your answers and **show all your work**

[10] 7. A turkey is put into an oven that has a constant temperature of 200°C. A thermometer embedded in the turkey registers its temperature. When the turkey is put into the oven, the thermometer reads 20°C, and 30 minutes later it reads 30°C. The turkey will be ready to each when the thermometer reads 80°C. How many minutes after being put into the oven will the turkey be ready to eat? Assume that the turkey's temperature satisfies Newton's law of cooling/heating.

$T(t) \sim$ temperature at time t (mins)

Newtons law of cooling : $\dfrac{dT}{dt} = k(T - T_s)$ where T_s is the temperature of the surroundings.

The solution to this differential equation is : Here $T_s = 200$.

$$T(t) = M + Ae^{kt}$$

$$T(t) = 200 + Ae^{kt}.$$

The conditions $T(0) = 20$ & $T(\frac{1}{2}) = 30$ are now used to determine A & k.

$T(0) = 20 \Rightarrow 20 = 200 + Ae^{k\cdot 0}$

$\Rightarrow 20 = 200 + A$

$\Rightarrow \boxed{A = -180}$

$T(30) = 30 \Rightarrow 30 = 200 - 180e^{k\cdot 30}$

$\Rightarrow \dfrac{-170}{-180} = e^{k\cdot 30}$

$\Rightarrow \ln(17/18) = k\cdot 30$

$\Rightarrow \boxed{k = \frac{1}{30}\ln(17/18)}$

$\therefore \boxed{T(t) = 200 - 180e^{\frac{1}{30}\ln(17/18)t}}$

time when $T(t) = 80$

$200 - 180e^{\frac{1}{30}\ln(17/18)t} = 8$

$e^{\frac{1}{30}\ln(17/18)t} = 12/18$

$\frac{1}{30}\ln(17/18)t = \ln(2/3)$

$\boxed{t = \dfrac{30\ln(2/3)}{\ln(17/18)}}$

[4] 8. (a) Show that Newton's Method applied to the equation $x^2 - a = 0$ yields the iterative formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

and thus provides a method for approximating the square root $\sqrt{a}$ which uses only addition and multiplication and division.

$$f(x) = x^2 - a$$
$$f'(x) = 2x$$

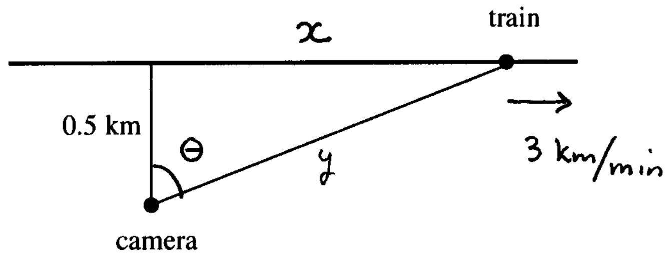
Newtons iterative formula :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - a}{2x_n} = \frac{2x_n^2 - x_n^2 + a}{2x_n}$$
$$= \frac{x_n^2 + a}{2x_n}$$
$$= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

[2] (b) Approximate $\sqrt{3}$ by taking $x_1 = 3/2$ and calculating x_2.

$$x_2 = \frac{1}{2}\left(x_1 + \frac{3}{x_1}\right)$$
$$= \frac{1}{2}\left(3/2 + 3/\tfrac{3}{2}\right)$$
$$= \frac{1}{2}\left(3/2 + 2\right)$$
$$= \frac{1}{2}\left(7/2\right)$$
$$= 7/4 .$$

[10] **9.** A high speed train is traveling at 3km/min along a straight track. The train is moving away from a movie camera which is located 0.5km from the track. The camera keeps turning so as to always point at the front of the train. How fast (in radians per minute) is the camera rotating when the front of the train is 1 km from the camera?



Know $\frac{dx}{dt} = 3$ km/min

Find $\frac{d\theta}{dt}$ when $y = 1$ km

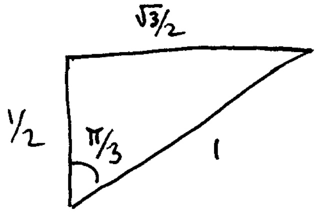
$$\tan\theta = \frac{x}{0.5} = 2x$$

$\frac{d}{dt}$

$$\sec^2\theta \, \frac{d\theta}{dt} = 2\,\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = 2\cos^2\theta \, \frac{dx}{dt}$$

When $y = 1$ km : $x = \frac{\sqrt{3}}{2}$ & $\theta = \pi/3$

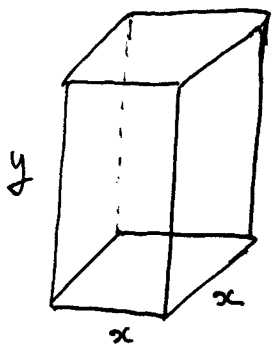


$$\left.\frac{d\theta}{dt}\right|_{y=1} = 2\cos^2(\pi/3)\cdot 3$$

$$= 2\left(\tfrac{1}{2}\right)^2 \cdot 3$$

$$= 3/2 \text{ rad/min}$$

[10] **10.** An open-topped box with a square base is to have volume 300 cm^3. The material for the bottom of the box costs 4 cents per cm^2 and the material for the sides of the box costs 2 cents per cm^2. What dimensions will minimize the total cost of the box?



$$x^2 y = 300 \leadsto y = \frac{300}{x^2}$$

Total cost :

$$C(x) = (4xy)\cdot 2 + x^2 \cdot 4$$

$$= 8xy + 4x^2$$

$$\boxed{C(x) = \frac{8(300)}{x} + 4x^2}$$

minimize $C(x)$ on $0 \leq x < \infty$

$$C'(x) = -\frac{2400}{x^2} + 8x$$

Critical points : $8x = \frac{2400}{x^2} \Rightarrow x^3 = 300$

$\Rightarrow x = \sqrt[3]{300}$

Does $C(x)$ have a minimum at $x = \sqrt[3]{300}$? Yes , since

$$\lim_{x\to 0^+} C(x) = \lim_{x\to 0^+} \frac{2400}{x} + 4x^2 = +\infty$$

$$C(\sqrt[3]{300}) = 8(300)^{2/3} + 4(300)^{2/3} = 12(300)^{2/3} \Leftarrow \text{minimum}$$

$$\lim_{x\to\infty} C(x) = \lim_{x\to\infty} \frac{2400}{x} + 4x^2 = +\infty$$

$\therefore$ Dimensions that minimize cost are $x = \sqrt[3]{300}$ & $y = \sqrt[3]{300}$.

[6] **11.** Find the derivative of

$$\frac{1}{1-x^2}$$

using the definition of the derivative. No marks will be given for the use of differentiation rules.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{1-(x+h)^2} - \frac{1}{1-x^2}}{h}$$

$$= \lim_{h\to 0} \frac{1-x^2 - 1 + (x+h)^2}{h\left[1-(x+h)^2\right]\left[1-x^2\right]}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h\left[1-(x+h)^2\right]\left[1-x^2\right]}$$

$$= \lim_{h\to 0} \frac{2x+h}{\left[1-(x+h)^2\right]\left[1-x^2\right]}$$

$$= \frac{2x}{(1-x^2)^2}$$

(Check using differentiation rules:

$$\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{d}{dx}(1-x^2)^{-1}$$

$$= -(1-x^2)^{-2}\cdot(-2x)$$

$$= \frac{2x}{(1-x^2)^2} \quad \checkmark$$

)

[4] **12.** Suppose $f(x)$ is a function that is differentiable for all x. Let $g(x)$ be the new function defined by $g(x) = f(x) + f(1 - x^2)$. Prove that $g'(c) = 0$ for some real number c in the interval $(0, 1)$.

$$g(0) = f(0) + f(1)$$

$$g(1) = f(1) + f(0)$$

By Mean value theorem there exists a c in (0,1) such that

$$g'(c) = \frac{g(1) - g(0)}{1 - 0}$$

$$= \frac{(\cancel{f(1)} + \cancel{f(0)}) - (\cancel{f(0)} + \cancel{f(1)})}{1}$$

$$= 0$$