Fall 2005, Mathematics 100 Final Exam Solutions

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1. (a) (4 points) $6x^2 - x - 1 = 0$

$$6x^2 - x - 1 = (3x+1)(2x-1) = 0$$

Thus $\boxed{x = -\frac{1}{3}}$ or $\boxed{x = \frac{1}{2}}$.

- If they factored correctly, but didn't solve for x, give 2 points.
- If they factored incorrectly 0 points.

(b) (4 points) $6x^2 - x - 1 \geq 0$

$$6x^2 - x - 1 = (3x+1)(2x-1) \geq 0$$

$\boxed{\{x \geq \frac{1}{2}\} \cup \{x \leq -\frac{1}{3}\}}$

- award one point for realizing that keeping the factored form is useful. (i.e. their solution has $\frac{1}{2}$ or $-\frac{1}{3}$ floating around).
- If they had a mistake in (a), and carried it thru to here, don't penalize them again.
- award three points for the correct solution. They may express their solution in any fashion, provided it is accurate.
- if they have extraneous expressions (or conflicting answers, e.g. the number line solution is correct, but they say $\frac{1}{2} \leq x \leq -\frac{1}{3}$, don't deduct any marks. Just circle the incorrect expression, incase they come in to look at their exams.
- award NO additional points if they just wrote $x \geq \frac{1}{2}$ & $x \geq -\frac{1}{3}$
- award one point if they considered only $x \geq \frac{1}{2}$ or $x \leq -\frac{1}{3}$.
- Subtract off one point if they used the wrong inequality, i.e. $x > \frac{1}{2}$.

(c) (5 points) $6e^{2x} - e^x - 1 = 0$

$$6e^{2x} - e^x - 1 = (3e^x + 1)(2e^x - 1) = 0$$

$e^x = \frac{1}{2} \implies \boxed{x = \ln\frac{1}{2}}$, while $e^x = -\frac{1}{3}$ gives no solution.

- award one point for factoring. Once again, if they have a mistake in (a) but carried it thru, don't penalize them.
- award two points for realizing $e^x = \frac{1}{2}$, $e^x = -\frac{1}{3}$.
- award one point for $x = \ln\frac{1}{2}$
- award one point for NOT including $x = \ln -\frac{1}{3}$.

(d) (7 points) $(\tan x + \sqrt{3})(\sin x + 2) = 0$

- award three points for stating explicitly that $\sin x \neq -2 \quad \forall x$
- award four points for correctly solving $\tan x = -\sqrt{3}$. i.e. $\boxed{x = -\frac{\pi}{3} + n\pi}$, where n is an integer.
- subtract two points if they forgot to include periodicity.
- If they just said $\tan x = -\sqrt{3}$ and $\sin x = -2$, give two points.

2. (a) (2 points) Define the *inverse of a function*?

Let $f(x)$ be a function. If $g(f(x)) = x$ and $f(g(x)) = x$, then $g(x)$ is the inverse of $f(x)$.

- the grading was generous
- one point award if they say that the function has to be 1:1 or if they said reflect about the line $y = x$ carefully.
- award two points if they say that $f(x)$ is a one-to-one function, the graph of the inverse function is the reflection of $f(x)$ about the line $y = x$.

(b) (3 points) Give an example of a function which has an inverse. Also, state its inverse. Be sure to explain why your example is invertible.

Many possible solutions. The easiest one is $f(x) = x \implies f^{-1}(x) = x$. $f(x)$ is an invertible function because it is one-to-one. (Satisfies horizontal line test)

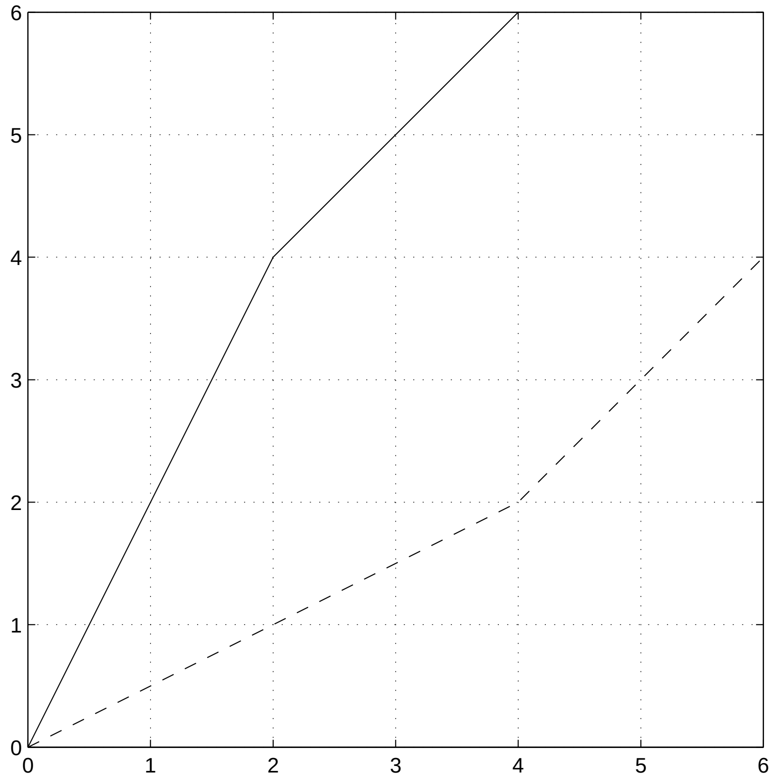
- subtract one point for no explanation of why function has an inverse.
- subtract one point for "lack of" or "bad" inverse.

(c) (2 points) Give an example of a function which does not have an inverse. Explain clearly why it is not invertible.

$f(x) = x^2$ does not have an inverse because it is not one-to-one.

- subtract two points if there are no explanations.

3. (a) (2 points) The graph of $f^{-1}(x)$ is shown below as a dotted line.



(b) (1 point) $f(1) = 2$.

(c) (1 point) $f(f(1)) = f(2) = 4$.

(d) (1 point) $f^{-1}(4) = 2$.

(e) (1 point) $f^{-1}(f(3)) = 3$.

4. (6 points) Sketch the graph of a function $f(x)$ which has *all* of the following properties.

   (i) $f(x)$ has exactly one root, $x = 1$.

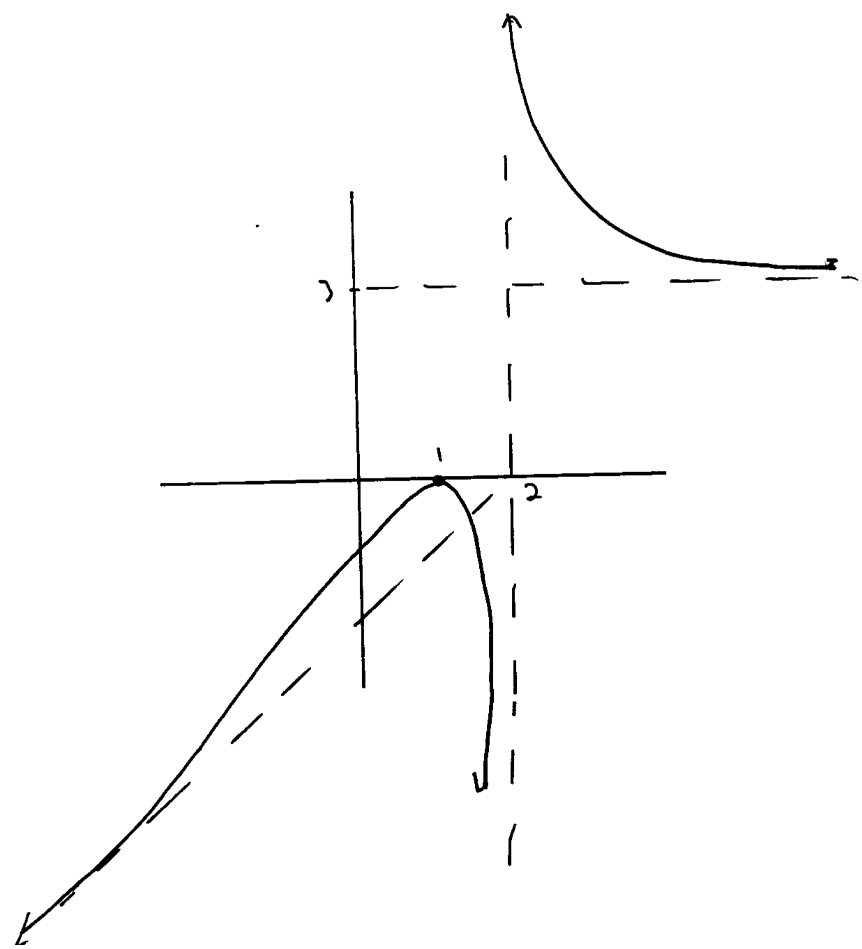
   (ii) As $x \to 2^+$, $f(x) \to \infty$.
   As $x \to 2^-$, $f(x) \to -\infty$.

   (iii) As $x \to \infty$, $f(x) \to 3$.
   As $x \to -\infty$, $f(x) \to x - 2$.

   (iv) $f(x)$ is continuous everywhere, except at $x = 2$.

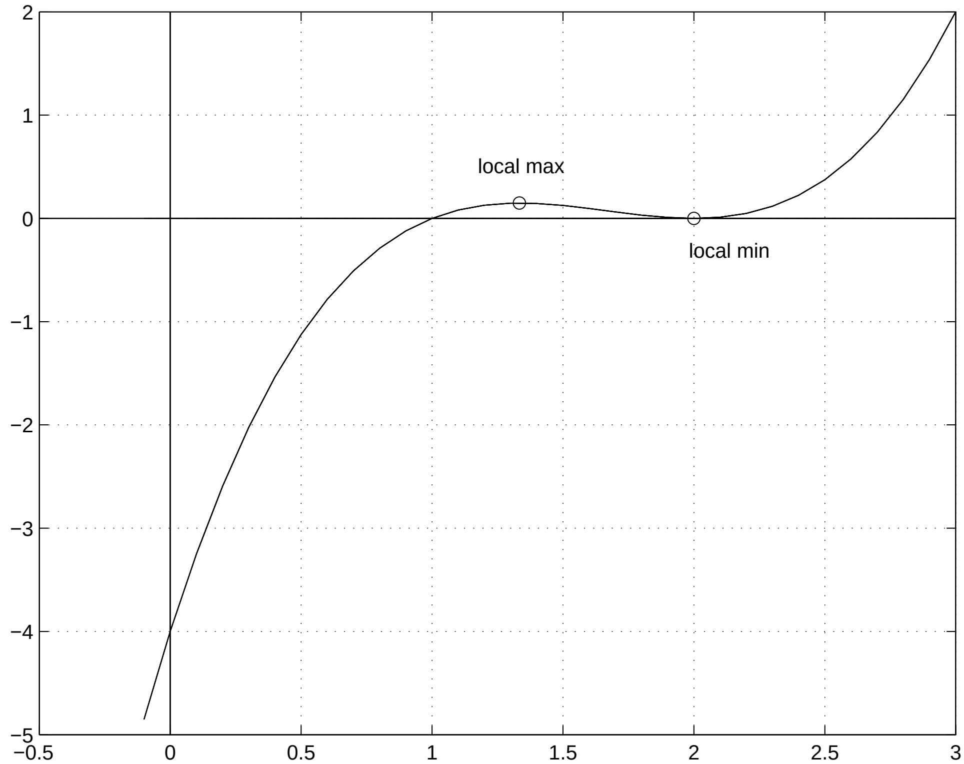
   (v) Verify that your graph satisfies (i).

Subtract off one point for each condition that is not satisfied. (Treat each limit as a seperate condition). Also, if the sketched graph was not a function, subtract 1 or 2 points depending on severity.



5. (a) (8 points) Sketch the function $f(x) = (x-1)(x^2-4x+4) = (x-1)(x-2)^2$.
   (i) (1 point) The function is neither odd nor even.
   (ii) (0.5 points) Domain: $\boxed{x \in R}$. (0.5 points) Range, $\boxed{f(x) \in R}$.
   (iii) (0.5 points) Function has no asymptotes.
   (iv) (0.5 points) y-intercept: $\boxed{f(0) = (-1)(4) = -4}$.
   (v) (0.5 points) $\boxed{x = 1}$ is a root of (0.5 points) $\boxed{\text{multiplicity 1}}$.
   (0.5 points) $\boxed{x = 2}$ is a root of (0.5 points) $\boxed{\text{multiplicity 2}}$.
   (vi) (1 point) local minima and maxima labeled below.

   Give two points for a plot that correctly represents information above, as found by the students.



(b) (8 points) Sketch the function

$$g(x) = -\cos\left[2\left(x - \frac{\pi}{8}\right)\right] = -\cos\left(2x - \frac{\pi}{4}\right)$$

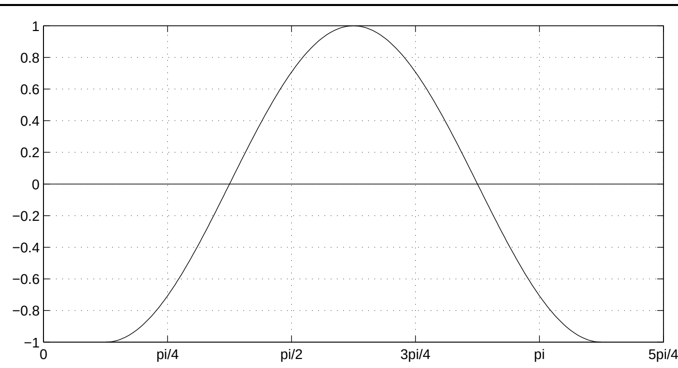
You should explicitly answer each of the following questions in your solution.

(i) Find the amplitude and period of $g(x)$.

(ii) Find an expression for all roots (x-intercepts) of $g(x)$.

- SCHEME 1: If the students attempt a series of transformations to go from $\cos x$ to $g(x)$, mark as follows. Starting with 8 points, subtract
  - two points for starting with the wrong function (i.e. $\sin x$ instead of $\cos x$)
  - two point for messing up the order of the horizontal expansion/compression and shift.
    - either shift to the right by $\frac{\pi}{4}$ followed by a horizontal compression by a factor of 2 or...
    - compress horizontally by a factor of 2, followed by a shift to the right by $\frac{\pi}{8}$.
  - one point for shifting in the wrong direction (should be a shift to the right)
  - one point for an incorrect horizontal compression. (should be a horizontal compression by a factor of 2).
  - one point for an incorrect reflection. (should be a reflection about the x-axis, or the line $y = 0$)
  - one point if their expression for the roots does not match the graph
  - one point if they forget to consider periodicity when giving an expression for the roots.
- SCHEME 2: If the students use the book technique of finding key points, assign marks as follows:
  - One point for the correct amplitude
  - One point for the correct period
  - Two points for the correct zeros (give only one point if they didn't consider periodicity)
  - Four points for a correct graph corresponding to their information above. No partial marks if graph doesn't match their solution.

Amplitude $= 1$
period $= \pi$,
xeros at $x = -\frac{\pi}{8} + n\frac{\pi}{2}$.



6. (6 points) Consider all pairs of real numbers whose difference is 8,
   (a) Determine the two numbers that have the minimum possible product.
   (b) What is the minimum product?

Let x and y be two real numbers.

- (1 point) $x - y = 8$
- (1 point) Product, $P = xy = (8 + y)(y)$
- (3 points) $P(y) = y^2 + 8y = (y + 4)^2 - 16 \implies$ vertex at (-4,-16) $\implies y = -4, x = 4$
- (1 point) minimum product = -16.
- Subtract 2 points if they had P(y) = 0 anywhere (that was not crossed off).

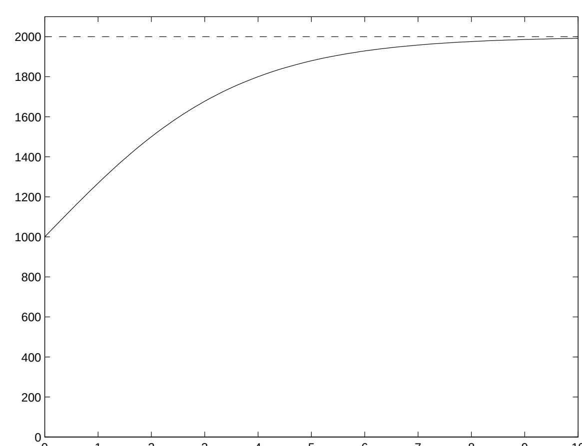
7. A small lake is stocked with a certain species of fish. The simplified logistic model below describes the population for time $t \geq 0$. The parameters B and k are positive constants.

$$P(t) = \frac{B}{1 + e^{-kt}}$$

   (a) (3 points) If initially the population is 1000, what limit does the population approach as $t \to \infty$?

- award 2 points for $P(0) = \frac{B}{1+1} = 1000 \implies B = 2000$.
- award 1 point for $P(t \to \infty) = B \quad (= 2000)$
- if they don't acknowledge that $P(t \to \infty) = B$, subtract a point.

   (b) Sketch a reasonably accurate graph of $P(t)$ for $t \geq 0$, showing explicitly any horizontal asymptotes.



(c) After 2 years, the population is 1500. What will the population be after 4 years?

- Subtract 1 point if P(0) = 0 is on the graph.
- award zero points of $P(0) \to \infty$
- full marks if their solution from part (a) is between 0 and a 1000, and they correctly sketch the graph and the asymptote.

(2 points) for either finding k correctly, or finding the correct expression for e^{-2k}. Give one or two points for some reasonable attempt to find k.

$$P(2) = \frac{2000}{1+e^{-2k}} = 1500 \implies \frac{4}{3} = 1 + e^{-2k}$$
$$\implies e^{-2k} = \frac{1}{3}$$
$$\implies -2k = -\ln 3 \implies k = \frac{\ln 3}{2}$$

(2 points) for attempting to plug in a value for k, regardless of whether they got the right answer above

$$P(4) = \frac{2000}{1+e^{-4k}} = \frac{2000}{1+(e^{-2k})^2} = \frac{2000}{1+\left(\frac{1}{3}\right)^2} = \frac{2000}{\frac{10}{9}} = 1800.$$

- for arthimetic errors (i.e. forgotten "-" sign, subtract 0.5 points.
- major arithmetic errors, subtract 1.
- if they had a different value for b in part (a), they did not lose marks in (c).

8. (4 points) Use a right angle triangle to find the exact value of

$$\sec\left(\sin^{-1}\frac{4}{5}\right)$$

- Let $x = \sin^{-1}\frac{4}{5}$.
- (1 point) This means that $\sin x = \frac{4}{5}$.
- (2 points) Thus $\cos x = \frac{3}{5}$
- (1 point) and $\sec x = \frac{1}{\cos x} = \frac{5}{3}$.

9. (a) Given that $\cos\alpha = \frac{1}{2}$, α lies in Quadrant IV, and $\sin\beta = -\frac{1}{3}$, β lies in Quadrant III, find $\sin(\alpha+\beta)$

- (2 points) Since $\cos\alpha = \frac{1}{2}$ and $\alpha \in Q4$, $\sin\alpha = -\frac{\sqrt{3}}{2}$.
- (2 points) Since $\sin\beta = -\frac{1}{3}$ and $\beta \in Q3$, $\cos\beta = -\frac{2\sqrt{2}}{3}$. (Don't deduct any points for $\frac{-\sqrt{8}}{3}$).
- (1 point)

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{-2\sqrt{2}}{3}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right) \\ &= \frac{2\sqrt{6}-1}{6}\end{aligned}$$

(b) Use your result from (a) to determine which Quadrant $(\alpha+\beta)$ must lie in.

$$\begin{aligned}\frac{3\pi}{2} <& \alpha < 2\pi \\ \pi <& \beta < \frac{3\pi}{2} \\ \frac{5\pi}{2} <& (\alpha+\beta) < \frac{7\pi}{2}\end{aligned}$$

(2 points) So $(\alpha+\beta)$ should lie in either quadrant II or III. Since $\sin(\alpha+\beta)$ is positive from part (a), we conclude that $(\alpha+\beta)$ lies in Quadrant II. (1 point)

- If they give some hand-wavy argument for why $(\alpha+\beta)$ has to lie in QII or QIII, that's okay too. Give 2 points
- If they just answer that $(\alpha+\beta)$ has to lie in QI or QII because $\sin(\alpha+\beta) > 0$, give just one point.

10. (5 points) Verify the identity

$$\sec(x+y) = \frac{\sec x \sec y}{1 - \tan x \tan y}$$

Many ways to solve this identity. Good application of some important/relevant identities, give one or two points. Correct proof, full 5 points.

BONUS

(6 points) They may submit only one of question B1 and B2. Mark only the one that is clearly attemped/circled.

B1. Consider two periodic functions, f, which has period 2π, and g, which has period 3π. What is the period of $f \circ g$?

> This question was actually more challenging than I had initially planned. Embarassingly, I had the wrong answer/concept when I was designing this question. The solution now, is of course painfully obvious.
>
> $$f(g(x)) = f(g(x + 3\pi)) \implies (f \circ g)(x) = (f \circ g)(x + 3\pi)$$
>
> Since $g(x)$ is 3π periodic. If anyone in Math 100 used this explanation, award the full 6 points for this test.

In actuality, one needs to also state that if the period of $g(x)$ is some integer multiple of the period of $f(x)$, say k, then the period of $(f \circ g)(x)$ will be the period of $g(x)$ divided by k. Since 3π is not an integer multiple of 2π, the period of $(f \circ g)(x)$ in this problem is 3π.

B2. Prove the identity

$$\ln\left(\cos 4t + 8\sin^2 t \cos^2 t\right) = 0$$

> There are also many ways to solve this identity. For good application of some important/relevant identities, give one or two points. Correct proof, full 6 points.