Fall 2005, Mathematics 100 Final Exam

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Last Name: _____ First Name: _____

Signature: _____ Student Number: _____

Instructions (Please Read)

- This exam has 16 pages, including this cover page.
- Read the questions carefully, show all of your work and present organized solutions. If we do not understand what you are doing, you will received *zero marks*. You *must* give reasons for your answers.
- The use of any kind of calculator is considered *academic dishonesty*.
- No notes or textbooks may be used.
- Some answers may contain radicals of the form $\sqrt{2}$ or logarithms of the form $\log 3$. These should be simplified where possible, however, a decimal expansion is not required.
- If you require more room for your solutions, you may use the backs of the pages. However, cross out any rough work that you do not wish to be graded.
- The last page of this exam is a blank page that you may detach for rough calculations. No work on this page will be graded. You MUST transfer any work that you would like to be graded to the exam booklet proper.
- The second last page of this exam contains the graph of a unit circle and useful identities. You may also detach this page for convenience.
- With the exception of the last two pages, the remainder of the booklet must be kept intact.
- The number of points for each question is listed in the left margin.
- The last section consists of two bonus questions. You may submit your solution to only one of these questions for additional marks.
- Nobody will be allowed to leave the examination room during the first hour of the exam. Latecomers will not be admitted to the examination room after the first hour of the exam.
- To avoid disturbing others, nobody will be allowed to leave the examination room during the last 30 minutes of the exam.

Question	Mark	Total Marks	Question	Mark	Total Marks
1		20	7		9
2		7	8		4
3		6	9		8
4		6	10		5
5		16	Bonus Question		6
6		6	Total		93 / 87

1. (20 points) For each of the following, find *all* real values of x that satisfy each of the given equations or inequalities.

   (a) $6x^2 - x - 1 = 0$

   (b) $6x^2 - x - 1 \geq 0$

(c) $6e^{2x} - e^x - 1 = 0$

(d) $(\tan x + \sqrt{3})(\sin x + 2) = 0$

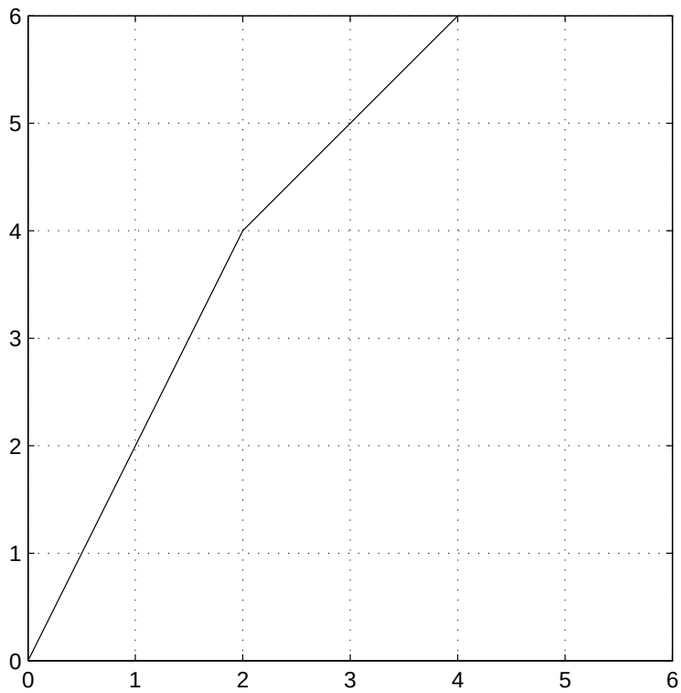
2. (7 points) The following question tests your knowledge about an inverse function.

   (a) Define the *inverse of a function*?

   (b) Give an example of a function which has an inverse. Also, state its inverse. Be sure to explain why your example is invertible.

   (c) Give an example of a function which does not have an inverse. Explain clearly why it is not invertible.

3. (6 points) Consider the following plot of $f(x)$.



(a) Overlay the graph of $f^{-1}(x)$ accurately on the plot above.

(b) Evaluate $f(1)$.

(c) Evaluate $f(f(1))$.

(d) Evaluate $f^{-1}(4)$.

(e) Evaluate $f^{-1}(f(3))$.

4. (6 points) Sketch the graph of a function $f(x)$ which has *all* of the following properties. (Note, you are not being asked to give an explicit formula for f.)

   (i) $f(x)$ has exactly one root, $x = 1$.

   (ii) As $x \to 2^+$, $f(x) \to \infty$.
   As $x \to 2^-$, $f(x) \to -\infty$.

   (iii) As $x \to \infty$, $f(x) \to 3$.
   As $x \to -\infty$, $f(x) \to x - 2$.

   (iv) $f(x)$ is continuous everywhere, except at $x = 2$.

   (v) Verify that your graph satisfies (i).

5. (a) (8 points) Sketch the function $f(x) = (x-1)(x^2-4x+4)$. You should explicitly answer each of the following questions in your solution.

   (i) Is the function odd, even or neither?

   (ii) What are the domain and range of the function?

   (iii) Does the function have any asymptotes (vertical, horizontal or slant)?

   (iv) What is the y-intercept?

   (v) Find all roots (x-intercepts) and their multiplicities. If the function has no roots, state this explicitly, along with your reasoning.

   (vi) Label any local minima or maxima on the graph of $f(x)$.

(b) (8 points) Sketch the function

$$g(x) = -\cos\left[2\left(x - \frac{\pi}{8}\right)\right] = -\cos\left(2x - \frac{\pi}{4}\right)$$

You should explicitly answer each of the following questions in your solution.

(i) Find the amplitude and period of $g(x)$.

(ii) Find an expression for all roots (x-intercepts) of $g(x)$. (Don't forget about periodicity). If there are none, state so explicitly.

6. (6 points) Consider all pairs of real numbers whose difference is 8,
   (a) Determine the two numbers that have the minimum possible product.
   (b) What is the minimum product?

7. (9 points) A small lake is stocked with a certain species of fish. The simplified logistic model below describes the population for time $t \geq 0$. The parameters B and k are positive constants.

$$P(t) = \frac{B}{1 + e^{-kt}}$$

(a) If initially the population is 1000, what limit does the population approach as $t \to \infty$?

(b) Sketch a reasonably accurate graph of $P(t)$ for $t \geq 0$, showing explicitly any horizontal asymptotes.

(c) After 2 years, the population is 1500. What will the population be after 4 years? (Note that using the properties of the exponential function, you are expected to obtain an exact numerical answer, with no unevaluated exponentials or logarithms).

8. (4 points) Use a right angle triangle to find the exact value of

$$\sec\left(\sin^{-1}\frac{4}{5}\right)$$

9. (8 points) Given that $\cos\alpha = \frac{1}{2}$, α lies in Quadrant IV, and $\sin\beta = -\frac{1}{3}$, β lies in Quadrant III, find

   (a) $\sin(\alpha + \beta)$

   (b) Use your result from (a) to determine which Quadrant $(\alpha + \beta)$ must lie in.

10. (5 points) Verify the identity

$$\sec(x+y) = \frac{\sec x \sec y}{1 - \tan x \tan y}$$

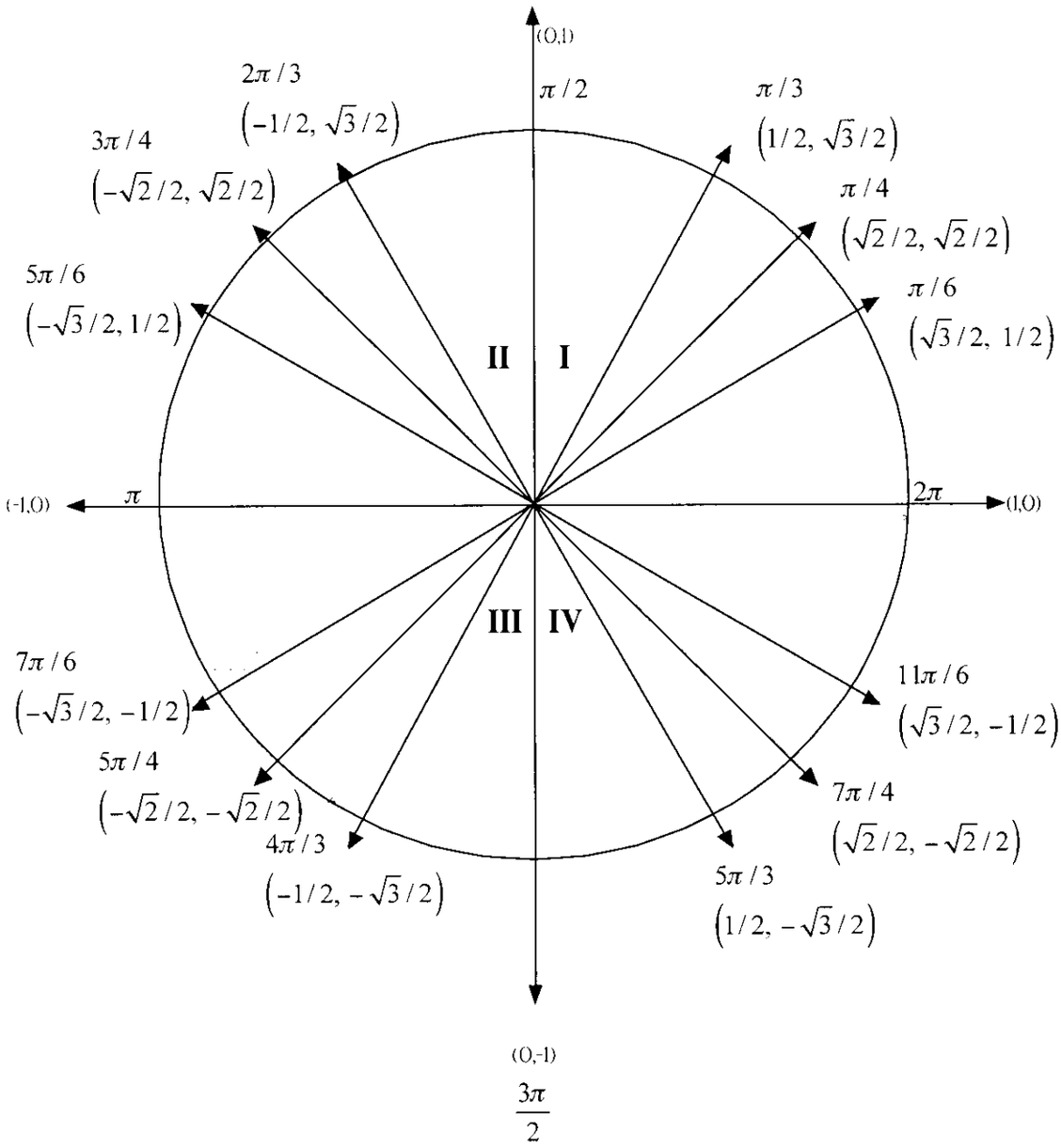
BONUS

(6 points) You may submit only one of question B1 and B2. Should you work on more than one of the questions, circle the question number of the one which you would like to be graded.

B1. Consider two periodic functions, f, which has period 2π, and g, which has period 3π. What is the period of $f \circ g$?

B2. Prove the identity

$$\ln\left(\cos 4t + 8\sin^2 t \cos^2 t\right) = 0$$



Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x.$$

Reciprocal Identities

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x}.$$

Sum & Difference Identities

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \quad \sin(x-y) = \sin x \cos y - \cos x \sin y,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \quad \cos(x-y) = \cos x \cos y + \sin x \sin y.$$

Double Angle Identities

$$\sin(2x) = 2 \sin x \cos x, \quad \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

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