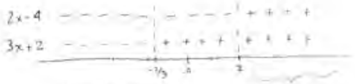
(1) [Marks: 4] Express $f(x) = \left|\frac{2x-4}{3x+2}\right|$ as a piecewise function.



$\frac{2x-4}{3x+2}$

$$f(x) = \begin{cases} \frac{2x-4}{3x+2} & x < -\frac{2}{3} \\ -\frac{2x-4}{3x+2} = \frac{4-2x}{3x+2} & \frac{2}{3} \le x < 2 \\ \frac{2x-4}{3x+2} & 2 \le x \end{cases}$$

(2) [Marks: 4] Consider the following functions:

$D(p)$ is the distance (in kilometres) one can travel on a train by buying a ticket worth p dollars

$T(d)$ is the time taken (in hours) to travel on a train a distance of d kilometres

$W(t)$ is Bill's weekly salary (in dollars) where t is the week of the year (1 - 52)

(a) What is the meaning of the following function: $T \circ D$?

$(T \circ D)(x) = T(D(x))$; so input x must be dollars (for D) and output must be time (from T)

$\rightarrow (T \circ D)(p)$ is time taken (in hours) travelling on a train when you spend p dollars (for a ticket).

(b) What combination of two or more of these 3 functions will tell you <u>how far Bill</u> can travel by train <u>each week</u> if he uses his entire weekly salary?

output

input

input : week   output ; distance

$$\xrightarrow[t]{\text{week}} \boxed{\;F\;} \xrightarrow[\text{Bill can travel}]{\text{distance}}$$

what is function F ?

functions that have input week : W ← inner function

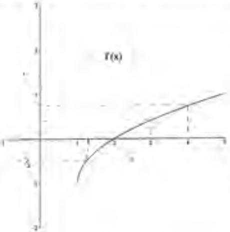
function that have output distance : D ← outer function

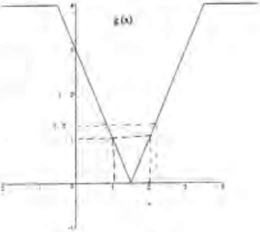
only choices for the composition

$(D \circ W)(t) = D(W(t))$: $\underset{\text{week}}{t} \rightarrow \underset{\substack{\text{dollars } d,\\ \text{Bill's salary}}}{W(t) = d} \rightarrow \underset{\text{distance travelled}}{D(d)}$

So $D \circ W$ is distance travelled on Bill's weekly salary.

(3) [Marks: 9] Below are the graphs of two functions $f(x)$ and $g(x)$. The domain of $f(x)$ is $[1,\infty)$ and the range of $f(x)$ is $[-1,\infty)$. The domain of $g(x)$ is $(-\infty,\infty)$ and the range of $g(x)$ is $[0, 4]$.





(a) Using these graphs, determine (approximately) the value of $(f\circ g)(0)$ and $(g\circ f)(4)$. Explain your reasoning by referring to the graphs.

$(f\circ g)(0) = f(g(0))$; from graphs $g(0)=3$ and $f(3)\simeq 0.2$

So $(f\circ g)(0) = f(g(0)) = f(3) \simeq 0.2$

(b) Determine an (approximate) x for which $(f\circ g)(x) = -1/2$. Explain your reasoning by referring to the graphs.

From graphs, $f(z) = -\frac{1}{2}$ for $z \simeq 1.3$

$g(x) = 1.3$ for $x \simeq \underline{0.9}$ or $\underline{2.3}$

(b) Determine the domain and range of $f\circ g$.

domain is x such that $g(x)$ in domain of f. Domain of f is $[1,\infty)$ so need $g(x) \geq 1$. From graph, $g(x)\geq 1$ if $x \leq 1$ or $x \geq 2$

So domain $f\circ g$ is $(-\infty,1] \cup [2,\infty)$.

range: for $x \in (-\infty,1]\cup[2,\infty)$, $g(x)$ gives values in $[1,4]$; call these values z. Then $f(z)$ gives values $[-1, 0.8]$ for these z.

So, range $f\circ g$ is $[-1, 0.8]$

(4) [Marks: 10] (a) Prove that $h(x) = \frac{|3x|}{1-2x^2}$ is not one-to-one.

$$h(-1) = \frac{|3(-1)|}{1-2(-1)^2} = \frac{|-3|}{1-2} = \frac{3}{-1} = -3$$

$$h(1) = \frac{|3(1)|}{1-2(1)^2} = -3$$

$h(-1) = h(1)$

(b) Prove that $f(x) = 3 + \sqrt{x^3 - 7}$ is one-to-one.

$$f(a) = f(b) \Rightarrow \cancel{3} + \sqrt{a^3 - 7} = \cancel{3} + \sqrt{b^3 - 7}$$

$$\rightarrow \sqrt{a^3-7} = \sqrt{b^3-7}$$

$$\rightarrow a^3 - 7 = b^3 - 7$$

(because if $\sqrt{x} = \sqrt{y}$ then $x = y$)

$$\rightarrow a^3 = b^3$$

$$\rightarrow a = b \quad \blacksquare$$

(c) Find $f^{-1}(x)$.

$$y = 3 + \sqrt{x^3 - 7}$$

$$y - 3 = \sqrt{x^3 - 7}$$

$$(y-3)^2 = x^3 - 7$$

$$x^3 = (y-3)^2 + 7$$

$$x = \sqrt[3]{(y-3)^2 + 7} \rightarrow f^{-1}(x) = \sqrt[3]{(x-3)^2 + 7}$$

(d) Verify that $(f^{-1} \circ f)(x) = x$.

$$f^{-1}(f(x)) = f^{-1}(3 + \sqrt{x^3 - 7}) = \left[\left(\left[3 + \sqrt{x^3-7}\right] - 3\right)^2 - 7\right]^{1/3}$$

$$= \left[\left(\sqrt{x^3 - 7}\right)^2 - 7\right]^{1/3} = \left[x^3 + 7 - 7\right]^{1/3} = \left[x^3\right]^{1/3} = x \quad \blacksquare$$

(5) [Marks: 10] Sketch **two** possible graphs of $p(x) = -2x^3(x-1)^3(x+2)^2$. Make use of the shape of the graph near the zeros, all intercepts, end behaviour, and the number of possible points where the tangent line is horizontal.

$p(x) = -2x^8 + \cdots$; end behavior $p(x) \to -\infty$ as $x \to \pm\infty$

degree $= 8 \Rightarrow$ at most 7 points where tangent line is horizontal.

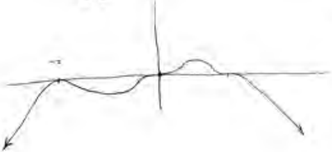
Behaviour near roots $0, 1, -2$;

$x=0$; $(x-1)^3 \sim (0-1)^3 = -1 < 0$, $(x+2)^2 > 0$ so $p(x) \sim kx^3$ where $k > 0$

$x=1$; $x^3 \sim 1^3 > 0$, $(x+2)^2 > 0$, so $p(x) \sim k(x-1)^3$ where $k < 0$

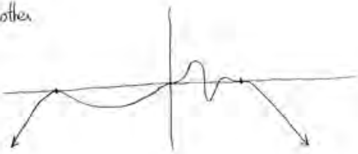
$x=-2$; $x^3 \sim (-2)^3 < 0$, $(x-1)^3 \sim (-2-1)^3 < 0$, so $p(x) \sim k(x+2)^2$ where $k < 0$

one possible graph



5 flat points

another



7 flat points

(6) [Marks: 9] Sketch the graph of $R(x) = \dfrac{-4x^2 - 8x + 12}{3x + 2}$. Find all intercepts, asymptotes, and determine how the graph approaches the asymptotes.

$$-4x^2 - 8x + 12 = -4(x-1)(x+3) = 0 \quad \text{at} \quad x = 1, -3$$

$$3x + 2 = 0 \quad \text{at} \quad x = -2/3$$

→ vertical asymptote $x = -2/3$; $-4x^2 - 8x + 12 > 0$

x near $-2/3$ but $x < -2/3$; $3x+2 < 0 \Rightarrow R(x) \to -\infty$

x near $-2/3$ but $x > -2/3$; $3x+2 > 0 \Rightarrow R(x) \to +\infty$

Horizontal asymptote : None since degree numerator > degree denominator

Slant asymptote :

$$\begin{array}{r|l} & -\frac{4}{3}x - \frac{16}{9} \\ 3x+2 & -4x^2 - 8x + 12 \\ & -4x^2 - \frac{8}{3}x \\ \hline & -\frac{16}{3}x + 12 \\ & -\frac{16}{3}x - \frac{32}{9} \\ \hline & 12 + \frac{32}{9} \end{array}$$

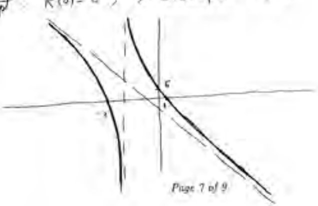
$$R(x) = \underbrace{-\frac{4}{3}x - \frac{16}{9}}_{\text{slant asymptote}} + \underbrace{\left(\frac{12 + 32/9}{3x+2}\right)}_{\to 0 \text{ as } x \to \pm\infty}$$

slant asymptote $y = -\frac{4}{3}x - \frac{16}{9}$

as $x \to +\infty$; $\dfrac{12 + 32/9}{3x+2} > 0$, $R(x)$ above asymptote

as $x \to -\infty$; $\dfrac{12 + 32/9}{3x+2} < 0$; $R(x)$ below asymptote

y-intercept $R(0) = 6$; x intercepts $1, -3$



(7) (Marks: 4) Given that 2 is a root of $p(x) = x^4 - 5x^3 + 7x^2 + 3x - 10$, find the complete factorization of $p(x)$.

$p(2) = 0 \Rightarrow p(x) = (x-2)q(x)$.

we see that $q(x) = (x^3 - 3x^2 + x + 5)$

Look for roots of $q(x)$; by trial and error we find

$q(-1) = 0$ so $q(x) = (x+1)h(x)$

where $h(x) = (x^2 - 4x + 5)$

Now use quadratic formula: $x = \frac{4}{2} \pm \frac{\sqrt{16-20}}{2}$

but discriminant is < 0 so no roots;

$x^2 - 4x + 5$ is irreducible.

$\rightarrow \quad p(x) = (x-2)(x+1)(x^2 - 4x + 5)$

(8) [Marks: 4] Find the partial fraction decomposition of $f(x) = \frac{-2}{3(x-1)(2x+1)(3x+2)}$

$$\frac{-2}{3(x-1)(2x+1)(3x+2)} = \frac{A}{3(x-1)} + \frac{B}{(2x+1)} + \frac{C}{3x+2}$$

multiply both sides by $3(x-1)(2x+1)(3x+2)$;

$$-2 = A(2x+1)(3x+2) + 3B(x-1)(3x+2) + 3C(x-1)(2x+1)$$

$x = 1$; $-2 = A(3)(5) \Rightarrow A = -2/5$

$x = -\frac{1}{2}$; $-2 = 3B(-\frac{3}{2})(-\frac{1}{2}) \Rightarrow B = -\frac{8}{9}$

$x = -\frac{2}{3}$; $-2 = 3C(-\frac{5}{3})(-\frac{1}{3}) \Rightarrow C = \frac{-18}{5}$

So

$$\frac{-2}{3(x-1)(2x+1)(3x+2)} = \frac{-2}{18x^3+3x^2-15x-6}$$

$$= \frac{-2}{15(x-1)} - \frac{8}{9(2x+1)} - \frac{18}{5(3x+2)}$$