MATH 100-D200 Instructor: R. Pyke Solutions

Midterm 2, November 9, 2007

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1. DO NOT LIFT UP THE COVER PAGE UNTIL INSTRUCTED.
2. Clearly explain your answer. No credit will be given for just writing down the answer.
3. If the answer space provided is not sufficient, write your answer on the back of the previous page.
4. Ordinary Scientific Calculators ONLY are allowed.
   NO GRAPHING CALCULATORS ALLOWED.
5. Copying someone else's test, or deliberately exposing written papers to the view of others is forbidden and will result in a score of zero and disciplinary action.

Question	Score	Max
1		9
2		8
3		~~[illegible]~~ 3
4		5
5		6
6		4
7		10
8		3
9		4
Total		~~[illegible]~~ 52

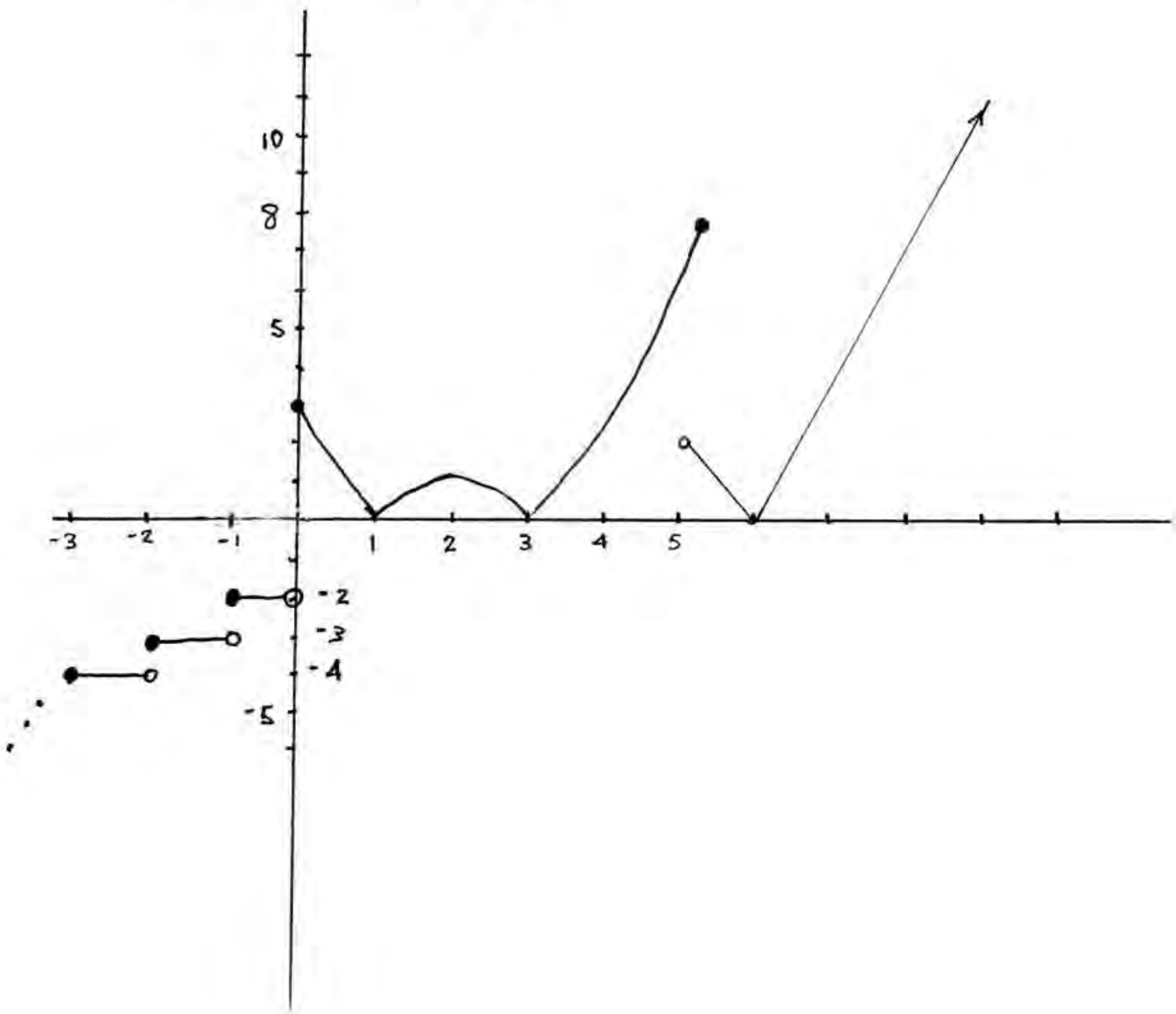
(1) [Marks: 9] Sketch the graph of the function $f(x)$. Remember that $\lfloor x \rfloor$ denotes the greatest integer function.

$$f(x) = \begin{cases} \lfloor x-1 \rfloor & x < 0 \\ |x^2-4x+3| & 0 \le x \le 5 \\ |3x-18| & 5 < x \end{cases}$$

$x^2-4x+3 = (x-2)^2-1 = (x-3)(x-1)$; vertex at $(2,-1)$

$\lfloor x-1 \rfloor$ is $\lfloor x \rfloor$ shifted right by 1

$|3x-18| = 3|x-6|$



(2) [Marks: 8] Consider the two functions

$$f(x) \;=\; \sqrt{\frac{x+1}{2x}}, \qquad g(x) \;=\; \frac{1}{x} \;+\; \sqrt{x}$$

(a) Find the domains of f and g.

f: $\dfrac{x+1}{2x} \geq 0$;

and $x \neq 0$

$- \quad - \quad + \quad + \quad +$ $\qquad x+1$

$-1 \qquad 0$

$- \quad - \quad - \quad + \quad +$ $\qquad 2x$

$\boxed{x \leq -1 \qquad 0 < x}$

need $(x+1)$ and $2x$ to have same sign

g: for $\frac{1}{x}$ need $x \neq 0$

for $\sqrt{x}$ need $x \geq 0$

$\Big\}$ $\boxed{x > 0}$

(b) Find $f \circ g$.

$$f(g(x)) = f\left(\frac{1}{x} + \sqrt{x}\right) = \sqrt{\frac{\frac{1}{x} + \sqrt{x} + 1}{2\left(\frac{1}{x} + \sqrt{x}\right)}}$$

$$g(f(x)) = g\left(\sqrt{\frac{x+1}{2x}}\right) = \sqrt{\frac{2x}{x+1}} + \underbrace{\sqrt{\sqrt{\frac{x+1}{2x}}}}_{\sqrt[4]{\frac{x+1}{2x}}}$$

(c) Find $g \circ f$.

(3) [Marks: ~~5~~ 3] Find two functions f, g, neither of which is the identity function $I(x) = x$, such that $F = f \circ g$ where

$$F(x) = x + 3 - \frac{2}{\sqrt{x+3}+1} + 1$$

$$f(x) = x^2 - \frac{2}{x+1} + 1 \quad , \quad g(x) = \sqrt{x+3}$$

or

$$f(x) = x - \frac{2}{\sqrt{x}+1} + 1 \quad , \quad g(x) = x + 3$$

(4) [Marks: 5] (a) Prove that $f(x) = \dfrac{x+1}{x-1}$ is one-to-one.

$$f(a) = f(b) \;;$$

$$\frac{a+1}{a-1} = \frac{b+1}{b-1}$$

$$\rightarrow (a+1)(b-1) = (b+1)(a-1)$$

$$\cancel{ab} - a + b = \cancel{ba} - b + a$$

$$2b = 2a$$

$$a = b \quad \checkmark$$

(b) Prove that $g(x) = \dfrac{x^2+1}{x^2-1}$ is not one-to-one.

take any number $a \neq 0$;

then $g(a) = g(-a)$

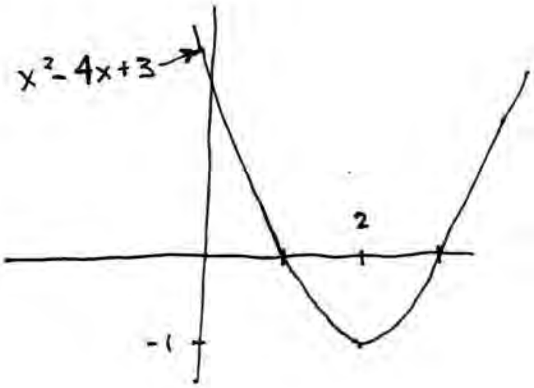
(5) [Marks: 6] Find $f^{-1}(x)$ for the function

$$f(x) = x^2 - 4x + 3 \qquad \text{with domain } (-\infty,\ 2]$$

$$y = x^2 - 4x + 3 \quad \rightarrow \quad x^2 - 4x + (3-y) = 0$$

use quadratic formula;

$$x = \frac{4 \pm \sqrt{16 - 4(3-y)}}{2}$$



$$= 2 \pm \frac{\sqrt{4+4y}}{2}$$

$$= 2 \pm \sqrt{1+y}$$

Here we choose the - root because the range of f^{-1} is $(-\infty, 2]$ (= domain f)

(6) [Marks:4] 5 is a zero of $f(x) = 4x^3 - 20x^2 - x + 5$. Find all the other zeros and give the complete factorization of $f(x)$.

$f(5) = 0 \Rightarrow x-5$ is a factor of $f(x)$;

$$\begin{array}{r|l} & 4x^2 - 1 \\ \hline x-5 & 4x^3 - 20x^2 - x + 5 \\ & 4x^3 - 20x^2 \\ \hline & 0 \quad\ 0 \quad -x + 5 \\ & \qquad\quad -x + 5 \\ \hline & \qquad\qquad 0 \end{array}$$

$$\left.\begin{aligned} f(x) &= (x-5)(4x^2-1) \\ &= 4(x-5)(x+\tfrac{1}{2})(x-\tfrac{1}{2}) \end{aligned}\right.$$

other roots; $+\frac{1}{2}, -\frac{1}{2}$

(7) [Marks: 10] Sketch the graph of $f(x) = \dfrac{x^3 + 2x}{x^2 - 1}$. Be sure to find all roots, asymptotes and how the graph approaches the asymptotes (<u>explain your reasoning</u>).

vertical asymptotes; $x^2 - 1 = 0 \rightarrow x = \pm 1$ (and $x^3 + 2x \neq 0$ for $x = \pm 1$)

near vertical asymptote

$$\left.\begin{array}{l} \text{for } x < -1, \ f(x) < 0 \\ x > -1, \ f(x) > 0 \end{array}\right\} \begin{array}{l} \downarrow \\ \uparrow \end{array}$$

$$\left.\begin{array}{l} x < 1, \ f(x) < 0 \\ x > 1, \ f(x) > 0 \end{array}\right\} \begin{array}{l} \downarrow \\ \uparrow \end{array}$$

No horizontal asymptote

$(f(x) \rightarrow \pm\infty \text{ as } x \rightarrow \pm\infty)$

Slant asymptote;

$$\begin{array}{r|l} & x \\ x^2-1 & x^3 + 2x \\ & x^3 - x \\ \hline & 3x \end{array}$$

$$\rightarrow f(x) = x + \frac{3x}{x^2-1}$$

$f(0) = 0$

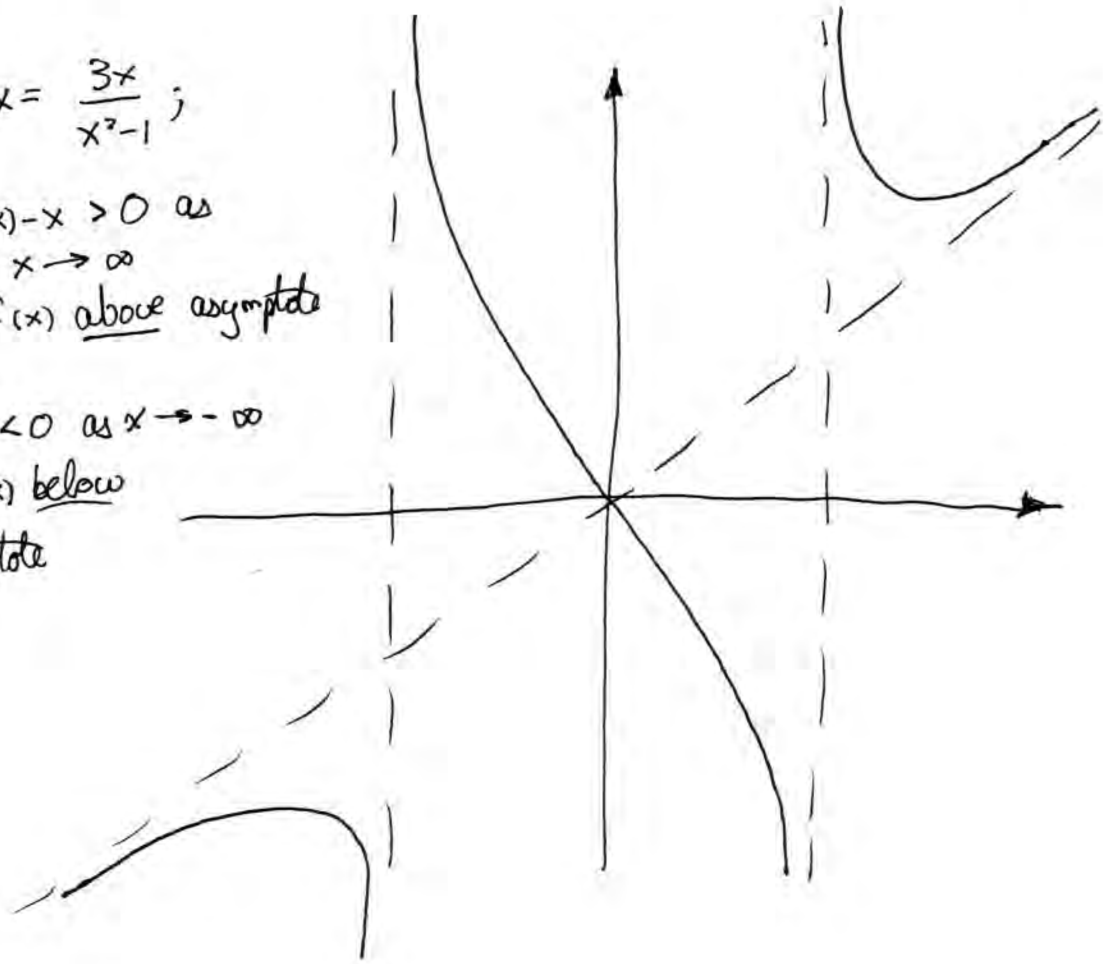
$f(x) - x = \dfrac{3x}{x^2-1}$;

So, $f(x) - x > 0$ as $x \rightarrow \infty$

$\Rightarrow f(x)$ <u>above</u> asymptote

$f(x) - x < 0$ as $x \rightarrow -\infty$

$\Rightarrow f(x)$ <u>below</u> asymptote



(8) [Marks: 3] Find all the roots (including complex roots) of $p(x) = x^3 + 2x^2 + 3x$.

$$p(x) = x\underbrace{(x^2+2x+3)}$$

$$x = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm \sqrt{2}\,i \quad , \quad i = \sqrt{-1}$$

$$\rightarrow p(x) = x(x+1-\sqrt{2}\,i)(x+1+\sqrt{2}\,i)$$

roots: $0, \ -1+\sqrt{2}\,i \ , \ -1-\sqrt{2}\,i$

(9) [Marks: 4] Find the partial fraction decomposition of $g(x) = \dfrac{1}{x(x-2)(2x-1)}$.

$$\frac{1}{x(x-2)(2x-1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{2x-1}$$

$$\rightarrow \quad 1 = A(x-2)(2x-1) + Bx(2x-1) + Cx(x-2)$$

$x=0$; $\quad 1 = A(-2)(-1) = 2A \rightarrow A = \frac{1}{2}$

$x=2$; $\quad 1 = B(2)(3) = 6B \rightarrow B = \frac{1}{6}$

$x=\frac{1}{2}$; $\quad 1 = C(\frac{1}{2})(-\frac{3}{2}) = -\frac{3}{4}C \rightarrow C = -\frac{4}{3}$