MATH 100 Final 07

(1) [Marks: 8] Solve the following inequalities. Express your answer in interval notation.

(a) $1 \le \dfrac{2x-6}{x-1} < 4$

$1 \le \frac{2x-6}{x-1}$

$\frac{2x-6}{x-1} - 1 \ge 0$

$\frac{2x-6-x+1}{x-1} \ge 0$

$\frac{x-5}{x-1} \ge 0$

$(-\infty, 1) \cup [5, \infty)$

$\frac{2x-6}{x-1} < 4$

$\frac{2x-6}{x-1} - 4 < 0$

$\frac{2x-6-4x+4}{x-1} < 0$

$\frac{-2x-2}{x-1} < 0 \quad \frac{x+1}{x-1} > 0$

$(-\infty, -1) \cup (1, \infty)$

(b) $\left|\dfrac{2-5x}{x}\right| \ge 5$

$\frac{2-5x}{x} \ge 5$ or $\frac{2-5x}{x} \le -5$

$\frac{2-5x-5x}{x} \ge 0$ $\quad$ $\frac{2-5x+5x}{x} \le 0$

$\frac{2-10x}{x} \ge 0$ $\quad$ $\frac{2}{x} \le 0$

$x < 0$

answer

~~$(-\infty, 0) \cup [\frac{1}{5}, \infty)$~~

~~$(-\infty, 0) \cup [\frac{1}{5}, \infty)$~~

$(0, \frac{1}{5}]$

(2) [Marks: 8] Solve the following equations (find all solutions).

(a) $| \, |x| - 4 \, | = 2$

$|x| - 4 = 2$

$|x| = 6$

$x = \pm 6$

$|x| - 4 = -2$

$|x| = 2$

$x = \pm 2$

(b) $\left| \dfrac{2x+1}{x-1} \right| = 3$

$\dfrac{2x+1}{x-1} = 3$

$2x + 1 = 3x - 3$

$x = 4$

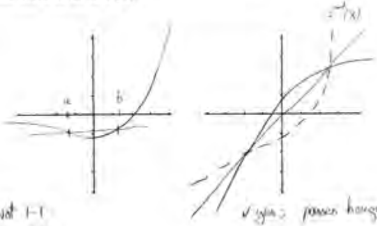
$\dfrac{2x+1}{x-1} = -3$

$2x + 1 = -3x + 3$

$5x = 2$

$x = \dfrac{2}{5}$

(3) [Marks: 6] For each of the graphs below, state whether the function is one-to-one or not (explain your reasoning). Also, if the function has an inverse, sketch the graph of the inverse on the same plot.



Not 1-1

$f(a) = f(b)$, $a \neq b$

✓ yes; passes horizontal line test

(4) [Marks: 6] Find the inverse of $f(x) = \dfrac{x-5}{2x+1}$ $= y$

$x - 5 = 2xy + y$

$x(1-2y) = y + 5$

$x = \dfrac{y+5}{1-2y} \rightarrow f^{-1}(x) = \dfrac{x+5}{1-2x}$

(5) [Marks: 9] (a) Let write out the formula for $f \circ g$ and $g \circ f$ where

$$f(x) = \frac{x^2}{1+\sqrt{x+1}}, \qquad g(x) = 2\cos(3x-1)$$

$$f \circ g = \frac{4\cos^2(3x-1)}{1+\sqrt{2\cos(3x-1)+1}}$$

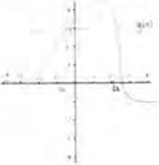
$$g \circ f = 2\cos\left(3\frac{x^2}{1+\sqrt{x+1}} - 1\right)$$

(b) Find two functions f and g so that $h = f \circ g$ where

$$h(x) = \frac{1}{2x+4} - 2\sqrt{2x+2} - 2$$

$$g = 2x+3$$
$$f = \frac{1}{x+1} - 2\sqrt{x-1}$$

(6) [Marks: 8] The graphs of two function $f(x)$ and $g(x)$ are given below. The domain of $f(x)$ is $(-3, 3]$. The domain of $g(x)$ is $(-\infty, \infty)$ and the range of $g(x)$ is $(-1, 4.5]$. Determine the domain and range of $f \circ g$. Explain your answer completely.



domain $f \circ g$ is $(-\infty, a] \cup [b, \infty)$. have to exclude $g(x) > 3$

range: - range f on $(-1, 3]$; is $(d, e]$

$f \circ g$

(7) [Marks: 8] Sketch the graph of the function $f(x) = \frac{x^2 - x - 6}{x - 5}$. Determine all asymptotes and intercepts, and be sure to determine how the graph approaches the asymptotes.

$$\begin{array}{r} x+4 \\ x-5\overline{)\,x^2-x-6} \\ \underline{x^2-5x} \\ 4x-6 \\ \underline{4x-20} \\ 14 \end{array}$$

$f = x + 4 + \frac{14}{x-5}$

slant asym $y = x + 4$

$f - y \to 0^+ \quad x \to \infty$; f above asymptote

$\quad\quad 0^- \quad x \to -\infty$; f below

vert asympt: $x = 5$

$x < 5$, near 5; $\frac{x^2-x-6}{x-5}$: $\frac{+}{-}$ } $f \to -\infty$ $x<5$

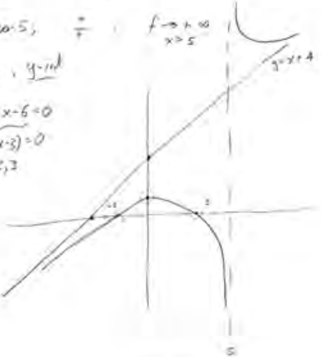
$x > 5$, near 5; $\frac{+}{+}$; $f \to +\infty$ $x > 5$

$f(0) = \frac{-6}{-5} = \frac{6}{5}$; y-int

$f(x) = 0$; $\underbrace{x^2 - x - 6}_{} = 0$

$(x+2)(x-3) = 0$

$x = -2, 3$



(8) [Marks: 4] Let $f(x) = 12x^3 + 16x^2 + 7x + 1$. Check that $f(-1/2) = 0$ and from this find the complete factorization of $f(x)$.

$f(-\frac{1}{2}) = 0$ ✓

so $f(x) = (x+\frac{1}{2})\, g(x)$:

$$
\begin{array}{r|l}
 & 12x^2 + 10x + 2 \\ \hline
x+\frac{1}{2} & 12x^3 + 16x^2 + 7x + 1 \\
 & \underline{12x^3 + 6x^2} \\
 & 10x^2 + 7x + 1 \\
 & \underline{10x^2 + 5x} \\
 & 2x + 1
\end{array}
$$

$$
\left\{
\begin{aligned}
f &= (x+\tfrac{1}{2})(12x^2 + 10x + 2) \\
&= (x+1)(6x^2 + 5x + 1) \\
&\qquad\quad (2x+1)(3x+1)
\end{aligned}
\right.
$$

or variations

$$
\left(\text{eg.}\quad
\begin{aligned}
&2(x+\tfrac{1}{2})(2x+1)(3x+1) \\
&(x+\tfrac{1}{2})(4x+2)(3x+1) \\
&(x+\tfrac{1}{2})(2x+1)(6x+2)
\end{aligned}
\right)
$$

(9) [Marks: 4] Find the complex roots of $2x^2 - 3x + 2$.

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{3 \pm \sqrt{9 - 4(2)(2)}}{4} \\
&= \frac{3 \pm \sqrt{-7}}{4} \\
&= \frac{3}{4} \pm \frac{\sqrt{7}}{4}\, i
\end{aligned}
$$

(10) [Marks: 8] (a) Write down the form of the partial fraction decomposition of the following rational function, <u>but do not solve for the coefficients</u>.

$$\frac{4x^2+4x-6}{(2x-3)^3(x^2-x+1)^2}$$

$$= \frac{A}{2x-3} + \frac{B}{(2x-3)^2} + \frac{C}{(2x-3)^3} + \frac{Dx+E}{(x^2-x+1)} + \frac{Fx+G}{(x^2-x+1)^2}$$

$1-4$

x^2-x+1

$x^2+x-1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$

$1+4$

(b) Find the partial fraction decomposition of the following rational function

$$\frac{3x}{x^2-16} = (x+4)(x-4)$$

$$= \frac{A}{x+4} + \frac{B}{x-4}$$

$$\rightarrow 3x = A(x-4) + B(x+4)$$

at $x=4$: $12 = 8B \rightarrow B = \frac{3}{2}$

at $x=-4$: $-12 = -8A \rightarrow A = \frac{3}{2}$

$$\rightarrow \frac{3x}{x^2-16} = \frac{3/2}{x+4} + \frac{3/2}{x-4}$$

$$\frac{3x^3}{x^2-16} = x^2-16 \overline{)\,3x^3}$$

$3x$

$3x^3-48x$

$48x$

$$= 3 + \frac{48x}{x^2-16}$$

(11) [Marks: 9] Determine the domains of the following functions.

(a) $y = \dfrac{1}{\sqrt{3 - 2x^2}} + \dfrac{x^3}{x+1}$

$$\Downarrow \qquad\qquad \Downarrow$$

$3 - 2x^2 = 0 \qquad x \neq -1$

$2x^2 < 3$

$x^2 < \frac{3}{2}$

$\Downarrow$

$-\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$ $\quad$ <u>and</u> $\quad x \neq -1$

(b) $y = \arcsin(x^2 - 1) - 2\arctan 2x$

$$\Downarrow \qquad\qquad \Downarrow$$

$-1 \leq x^2 - 1 \leq 1 \qquad \mathbb{R}$

$0 \leq x^2 \leq 2$

$-\sqrt{2} \leq x \leq \sqrt{2}$

(c) $f(x) = \dfrac{3}{\ln(x+2)} + 3^{-x}$

$$\swarrow \qquad\qquad \searrow$$

$x + 2 > 0 \qquad \mathbb{R}$

$\underline{x > -2}$

(12) [Marks: 8] Find the exact value of the given expression. Do not use your calculator for this; show all your work.

(a) $\sin\left(\frac{3\pi}{8}\right)$ $\quad \frac{1}{2}\ \frac{3\pi}{4}$; $\sin\frac{3\pi}{8} = \pm\sqrt{\frac{1-\cos\frac{3\pi}{4}}{2}}$

$$= \pm\sqrt{\frac{1+\sqrt{2}/2}{2}}$$

tak (+) sign

(b) $\tan(165°)$

$165° = 135° + 30°$

$$\tan 165° = \frac{\tan 135° + \tan 30°}{1 - \tan 135° \tan 30°}$$

$$= \frac{-1 + 1/\sqrt{3}}{1 + 1/\sqrt{3}}$$

(13) [Marks: 8] If $\tan x = 2$, $\pi < x < \frac{3\pi}{2}$, find (a) $\cos\left(\frac{x}{2}\right)$, and (b) $\csc(2x)$.

$\sec x = \pm\sqrt{\tan^2 x + 1} = \pm\sqrt{5}$; choose $(-)$

So, $\cos x = -\frac{1}{\sqrt{5}}$

$\rightarrow \boxed{\cos\frac{x}{2} = -\sqrt{\frac{1-1/\sqrt{5}}{2}}}$ $\quad \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$; $(-)$

$\sin x = -\sqrt{1-\frac{1}{5}} = -\sqrt{4/5}$

$\rightarrow \sin 2x = 2\sin x\cos x = \frac{4}{5}$

$\rightarrow \boxed{\csc 2x = \frac{5}{4}}$

(14) [Marks: 8] Find the period, x-intercepts, vertical asymptotes (if any), and sketch at least one cycle of the graphs of

(a) $y = \cot \pi x \ -1$

$\cot \pi x$

$\cot \pi x - 1$: shift down

(b) $y = -2\cos\left(2x + \frac{\pi}{3}\right)$

$\cos\left(2\left(x + \frac{\pi}{6}\right)\right)$.

$\cos 2x$

x int: $\frac{(2n-1)\pi}{4}$

odd mult $\frac{\pi}{4}$

$\frac{\pi}{4} \pm n\frac{\pi}{2}$

mult by -2 :

shift $\frac{\pi}{6}$ ←

period : π

x int ; $\frac{\pi}{12} \pm n\frac{\pi}{2}$

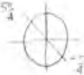
$\frac{\pi}{4} - \frac{\pi}{6} = \frac{(6-4)\pi}{24} = \frac{2\pi}{24} = \frac{\pi}{12}$

$\boxed{\frac{\pi}{12} \pm n\frac{\pi}{2}}$

(15) [Marks: 7] Find the indicated value.

(a) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right) \qquad = -\frac{\pi}{4}$

$\tan^{-1}x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



(b) $\cos\left(\overset{\theta}{\arctan(-2)}\right)$

$\tan\theta = -2, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan^2+1 = \sec^2; \quad 5 = \sec^2\theta$

$\to \cos^2\theta = \frac{1}{5} \to \cos\theta = \frac{1}{\sqrt{5}}$

(16) [Marks: 5] Write the following expression as an algebraic expression in x:

$\cos\left(\pi + \underbrace{\sin^{-1}(x)}_{\theta}\right)$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$t = \frac{\theta_1}{2} \quad \frac{\theta_2}{2}$ $+n\pi$

$\cos\left(\underbrace{\sin^{-1}x}_{\theta}\right)$

$\sin\theta = x$

$\to \cos\theta = \sqrt{1-x^2}$

Now, $\cos(\theta+\pi) = -\cos\theta$, so $\cos(\pi + \sin^{-1}(x)) = -\sqrt{1-x^2}$

(17) [Marks: 10] Solve the following equations (find all solutions).

(a) $1 - \sin x = 6 \sin^2 x$

$6z^2 + z - 1 = 0$; $z = \frac{-1 \pm \sqrt{1+24}}{12} = \frac{-1 \pm 5}{12} = -\frac{1}{2}, \frac{1}{3}$

$\sin x = -\frac{1}{2}$; $x = \frac{7\pi}{6}, \frac{11\pi}{6} \pm 2n\pi$

$\sin x = \frac{1}{3}$; $x = \theta_1, \theta_2 \pm 2n\pi$

$\sin 4x = 2\sin 2x \cos 2x$

(b) $8^{2-x} - 2 = 0$

$(2^3)^{2-x} = 2^1$

$\to 6 - 3x = 1$

$\to x = 5/3$

(c) $3 - \log_2 \sqrt{x^2 + 17} = 0$

$\log_2 \sqrt{(\ )} = 3$

$\to \sqrt{(\ )} = 2^3$

$\to x^2 + 17 = 2^6 = 64$

$\to x^2 = 47$

$x = \pm\sqrt{47}$

(18) [Marks: 12] (a) If $\log_3 x = a$, find

(i) $\log_3 \frac{x}{9}$ in terms of a.

$$\log_3 x - \log_3 9 = a - 2$$

(ii) $\log_{27} x$ in terms of a.

$$\underbrace{\log_{27} x}_{= b}$$

$\to x = 27^b$, but $x = 3^a$ $\Big\}$ $(3^3)^b = 3^a \to 3b = a \to \boxed{b = \frac{a}{3}}$

(b) If $3^x = b$, find

(i) $\left(\frac{1}{9}\right)^x$ in terms of b.

$$(3^{-2})^x = (3^x)^{-2} = b^{-2}$$

(ii) e^x in terms of b.

Let $3 = e^t$ $\quad (t = \ln 3)$

Then $3^x = (e^x)^t = b$

So $\boxed{e^x = b^{1/t}}$

(19) [Marks: 8] Sketch the graphs of the following functions.

(a) $f(x) = e^{-(x+1)} - 2$

e^{-x}

$e^{-(x+1)}$ shift left 1

$e^{-(x+1)} - 2$: shift down 2

(b) $f(x) = 3 - 2\log_2(x+2)$

$2\log_2(x)$

$2\log_2(x+2)$; shift left 2

$3 - 2\log_2(x+2)$

$-2\log_2(x+2)$

(20) [Marks: 6] A model for the population in a small community after t years is given by $P(t) = P_o e^{kt}$.

(a) If the initial population has doubled in 5 years, how long will it take to triple?

$P(5) = 2P_0 = P_0 e^{5k}$

$\rightarrow 5k = \ln 2 \rightarrow k = \frac{\ln 2}{5}$

Now $P(T) = 3P_0 = P_0 e^{\frac{\ln 2}{5}T}$

$\rightarrow \frac{\ln 2}{5} T = \ln 3 \rightarrow T = \frac{\ln 3}{\ln 2} \cdot 5 \approx 7.92$

(b) If the population in the community in part (a) is 10,000 after 3 years, what was the initial population?

$P(3) = 10{,}000 = P_0 e^{3k}$

$\rightarrow P_0 = 10{,}000 e^{-3k}$ (or $\ln(10{,}000) = \ln(P_0) + 3k$ etc)

$= 10{,}000 e^{-3\left(\frac{\ln 2}{5}\right)} \approx$