Specialized Gray Codes

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July 5, 2017

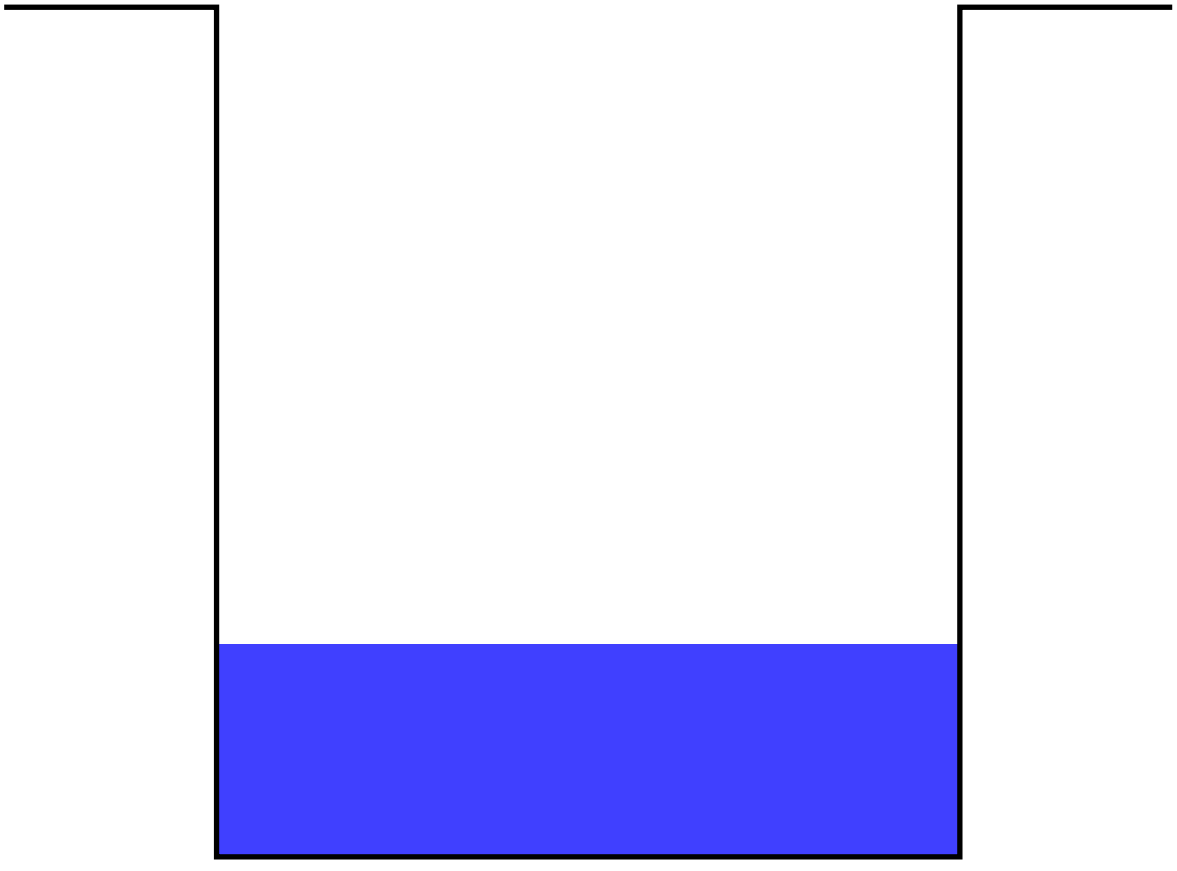
Support material for SFU Math Camp for Teachers

Frank Gray

Sam Beckett

Position-to-digital conversion issue

Simple example: Water level in a canal



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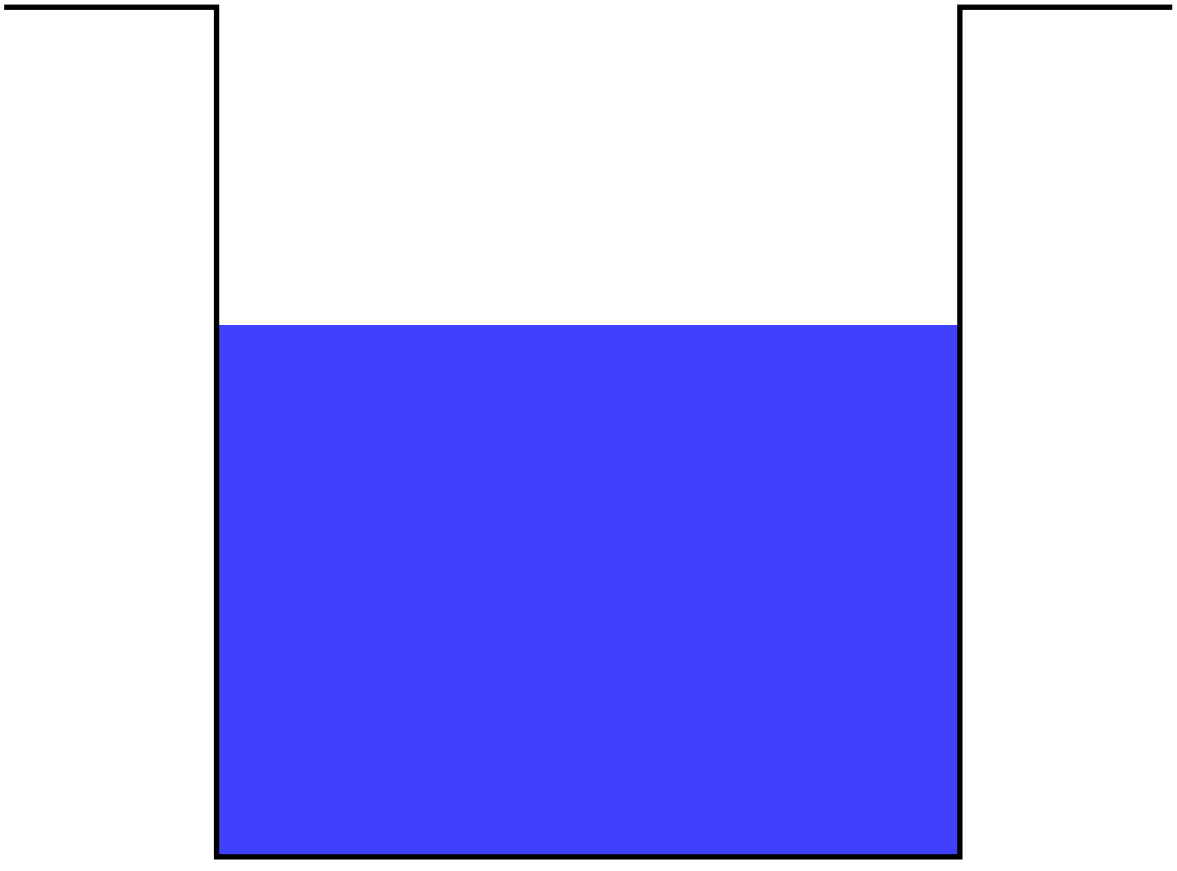

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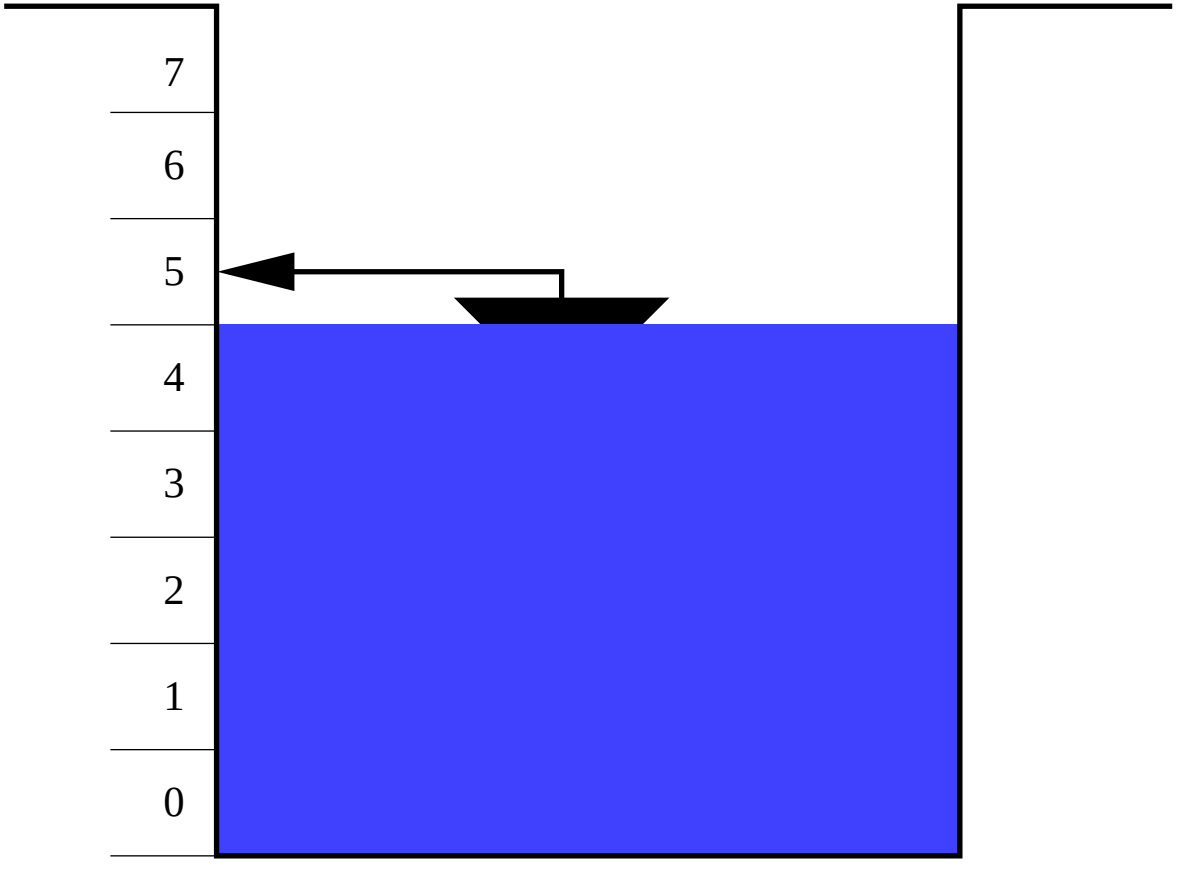

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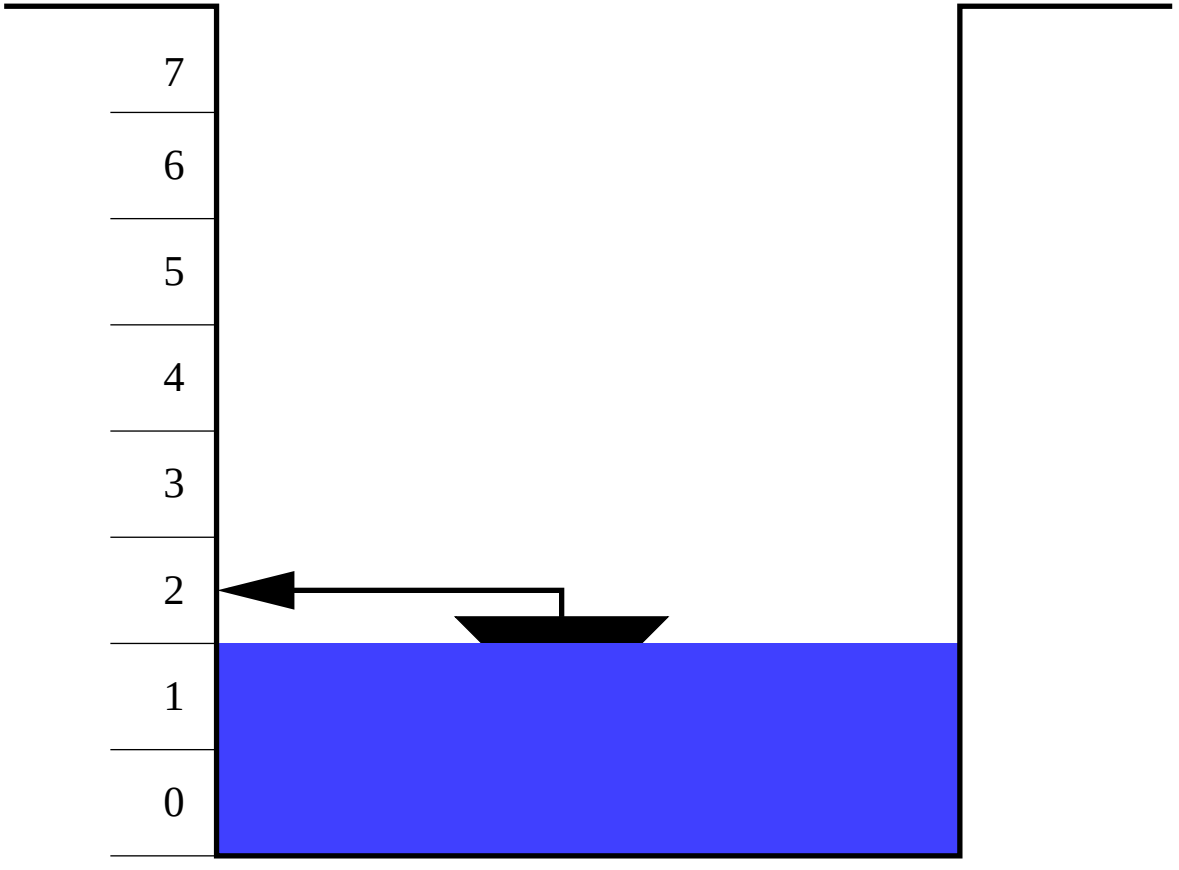

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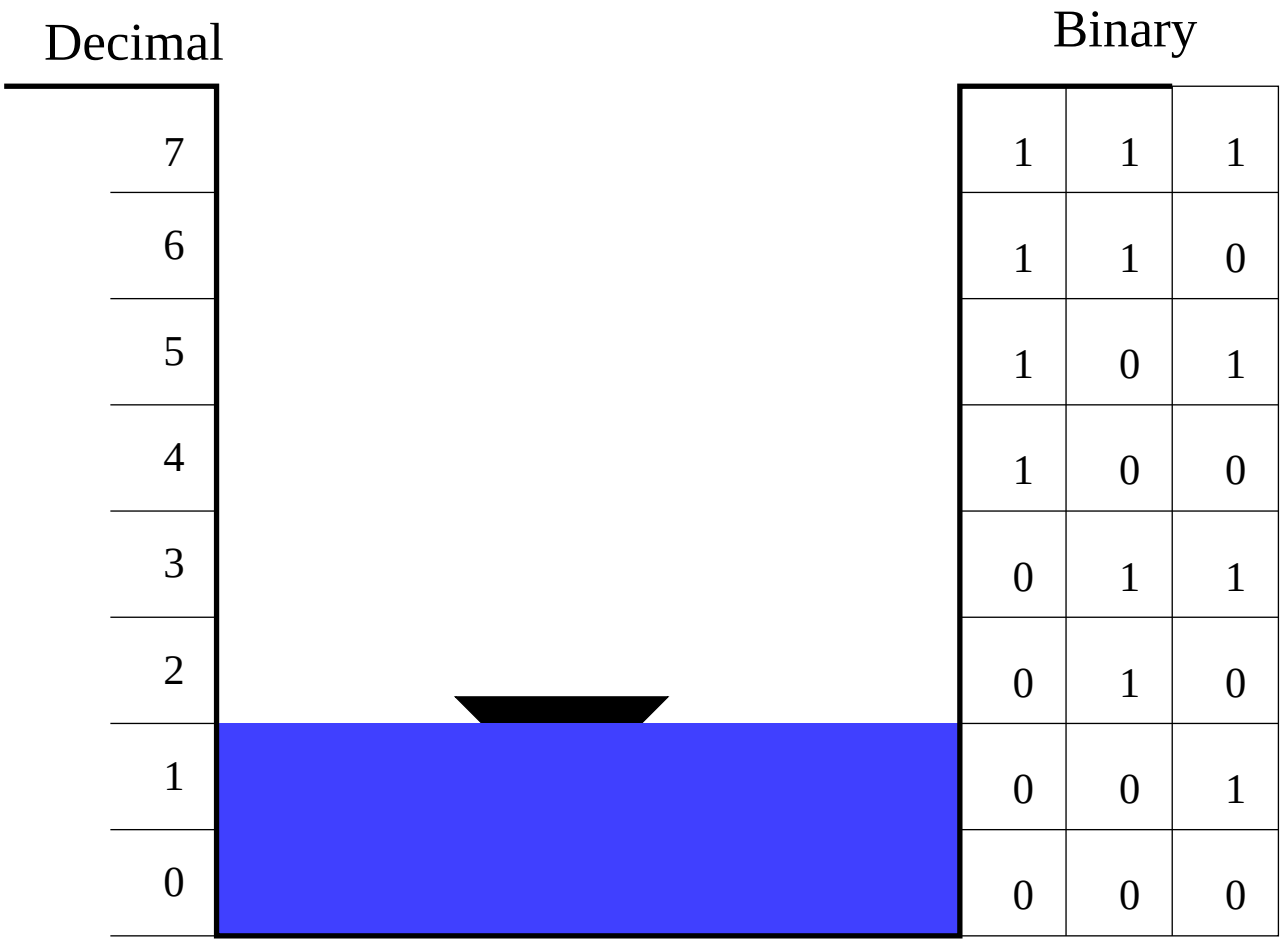
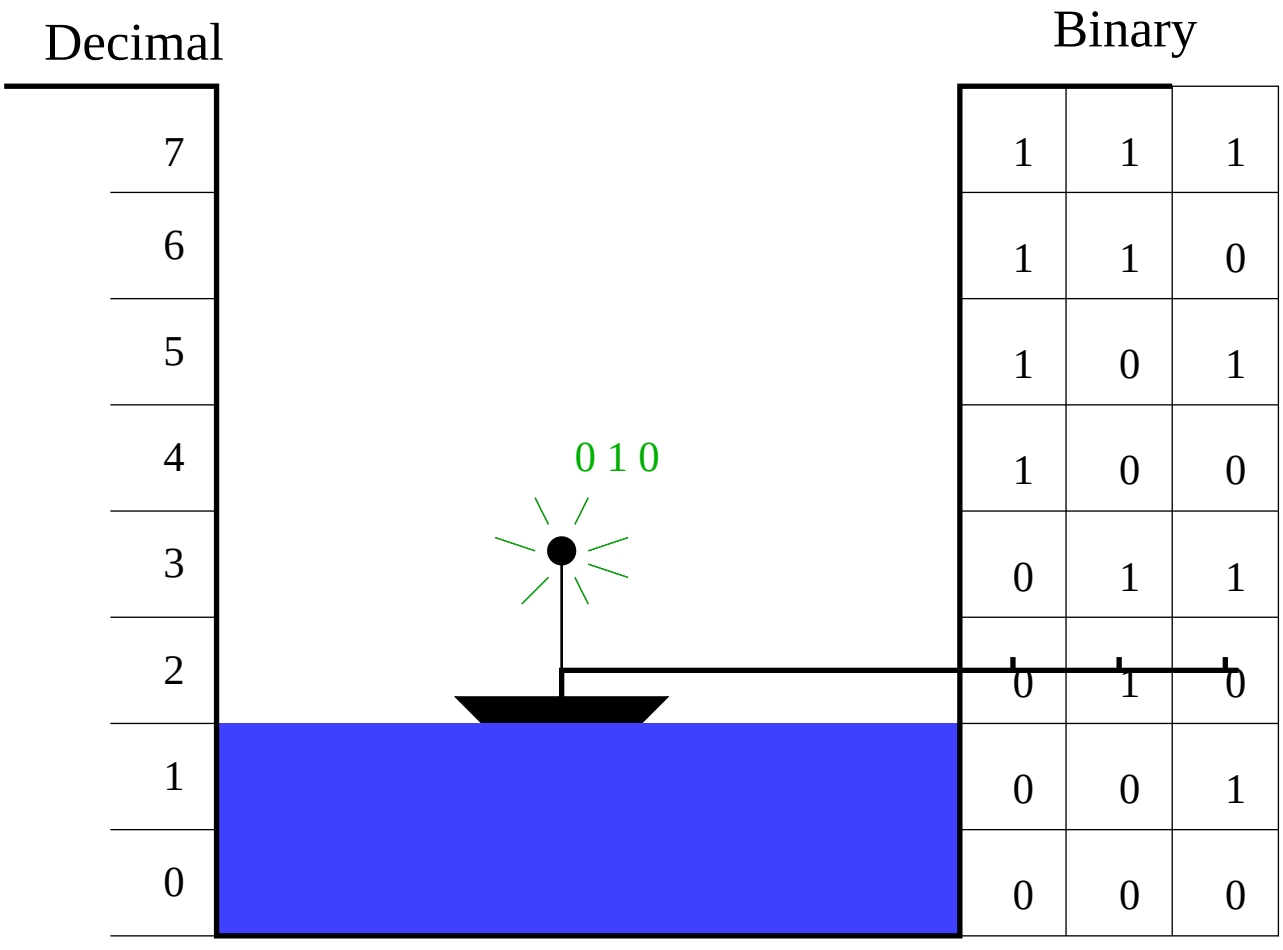


Base 2 counting for digital monitoring

Position-to-digital conversion issue

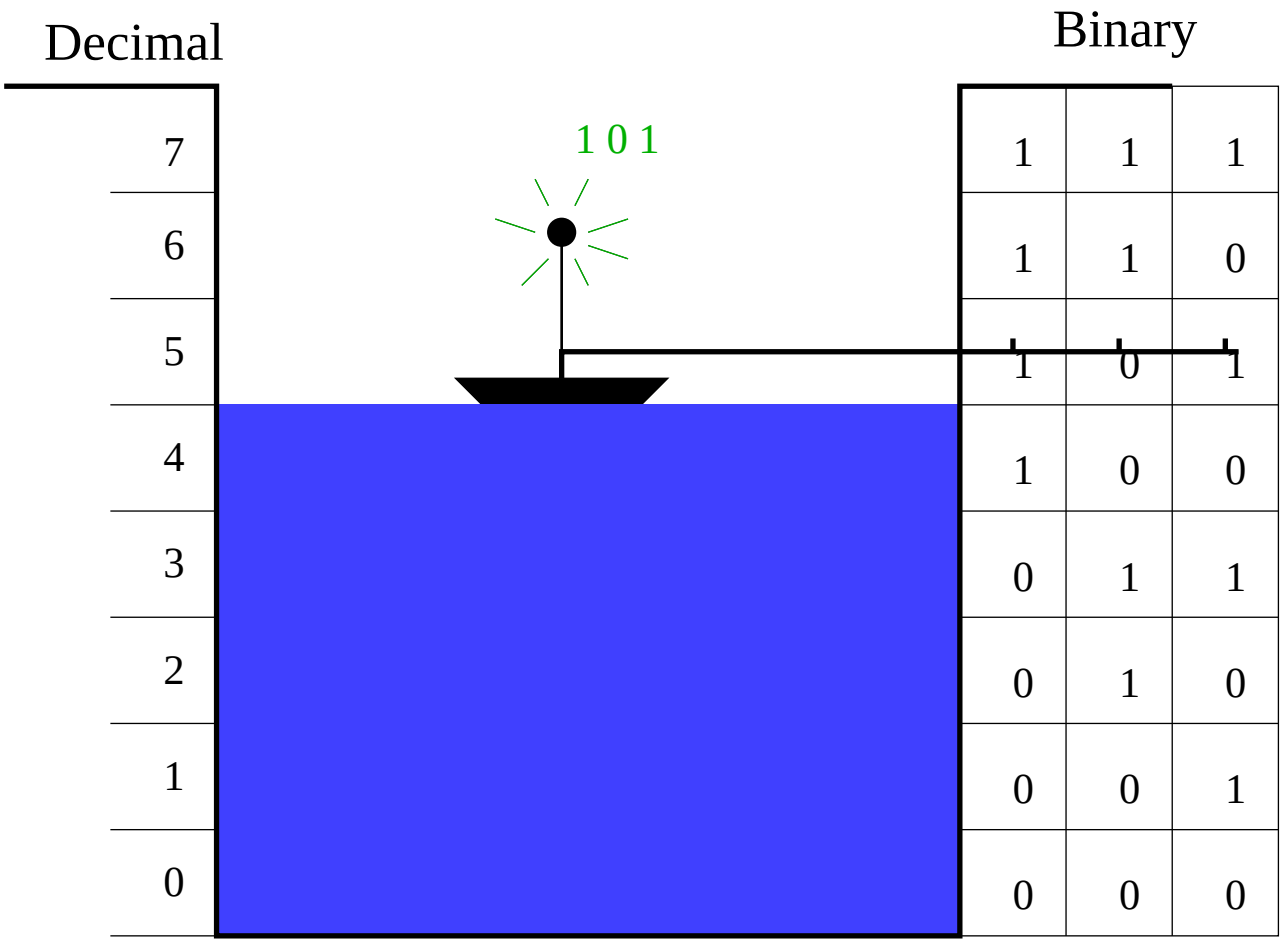
Simple example: Water level in a canal



Base 2 counting for digital monitoring

Position-to-digital conversion issue

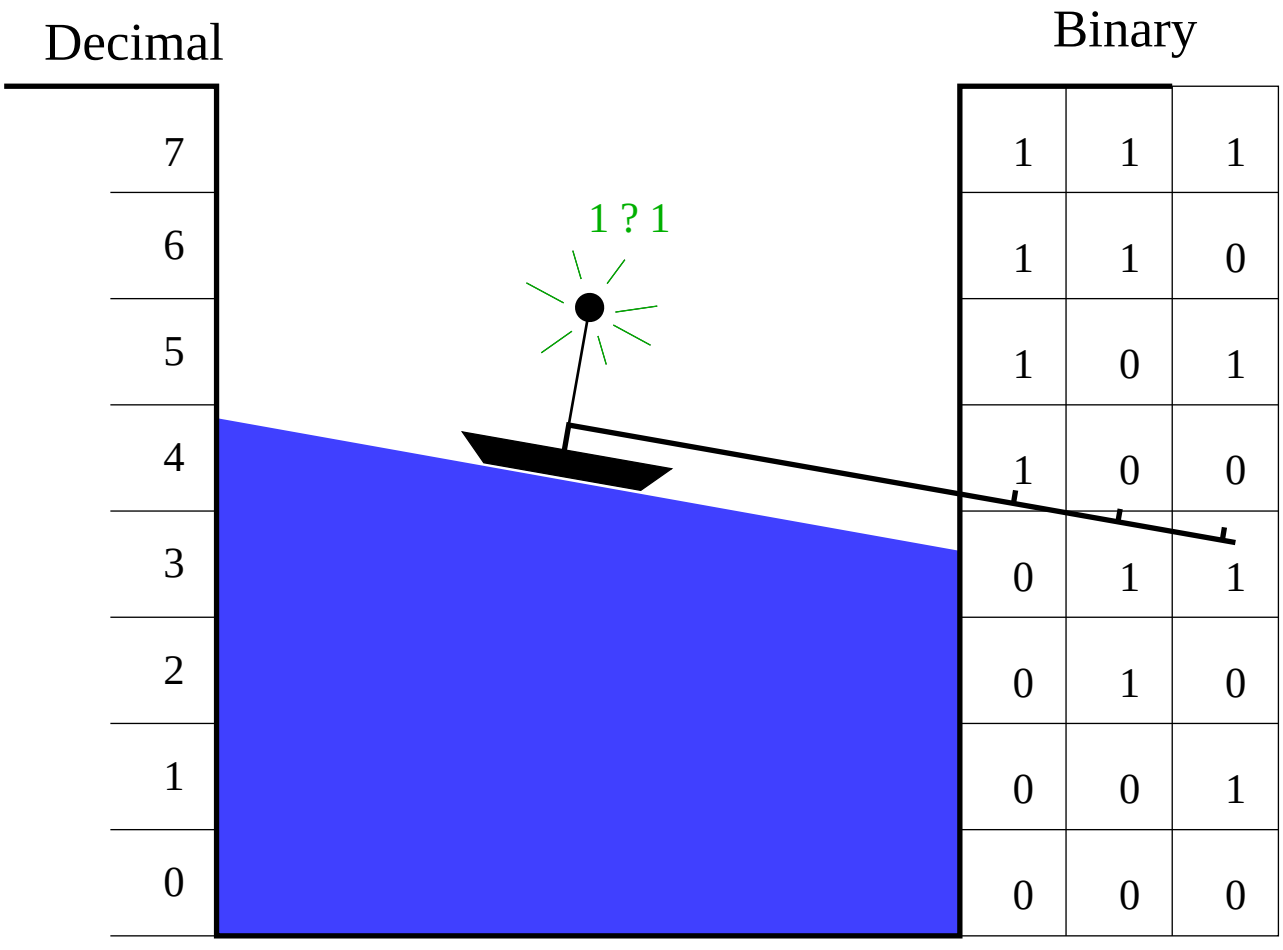
Simple example: Water level in a canal



Base 2 counting for digital monitoring

Position-to-digital conversion issue

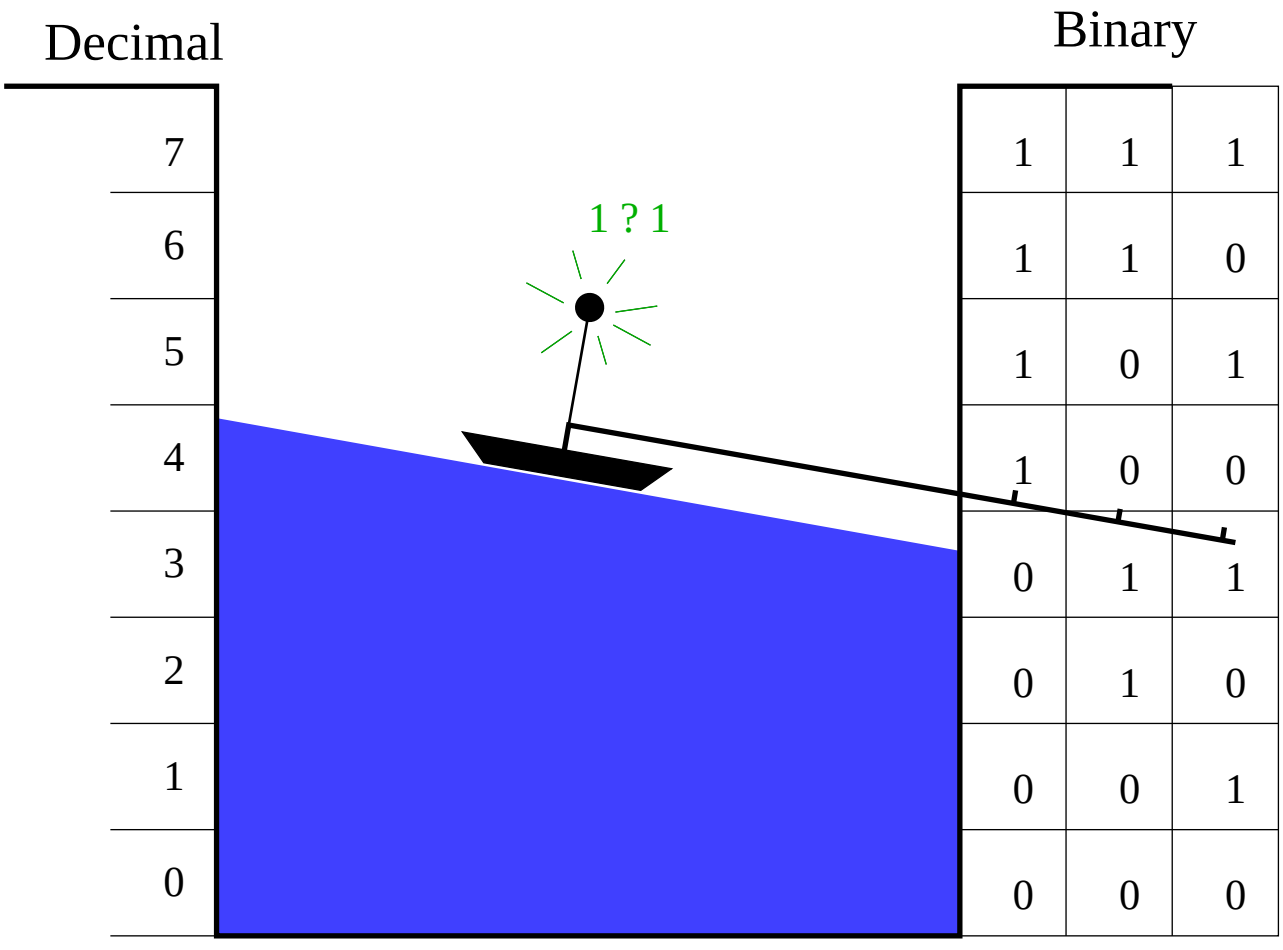
Simple example: Water level in a canal



Large position error!

Position-to-digital conversion issue

Simple example: Water level in a canal



Large position error!

Reason: Several digits change at once.

Frank Gray's solution - Reflected Binary Counting

Decimal
0
1
2
3
4
5
6
7
8
9

Frank Gray's solution - Reflected Binary Counting

Decimal
0
1
2
3
4
5
6
7
8
9
0
1
2
3
4
5
6
7
8
9

Frank Gray's solution - Reflected Binary Counting

Decimal	
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	10
11	11
12	12
13	13
14	14
15	15
16	16
17	17
18	18
19	19

Frank Gray's solution - Reflected Binary Counting

Decimal	
00	0
01	1
02	2
03	3
04	4
05	5
06	6
07	7
08	8
09	9
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	

Frank Gray's solution - Reflected Binary Counting

Decimal	
00	0
01	1
02	2
03	3
04	4
05	5
06	6
07	7
08	8
09	9
10	9
11	8
12	7
13	6
14	5
15	4
16	3
17	2
18	1
19	0

Frank Gray's solution - Reflected Binary Counting

Decimal	
00	00
01	01
02	02
03	03
04	04
05	05
06	06
07	07
08	08
09	09
10	19
11	18
12	17
13	16
14	15
15	14
16	13
17	12
18	11
19	10

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC
00	00
01	01
02	02
03	03
04	04
05	05
06	06
07	07
08	08
09	09
10	19
11	18
12	17
13	16
14	15
15	14
16	13
17	12
18	11
19	10

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary
00	00	00
01	01	01
02	02	10
03	03	11
04	04	
05	05	
06	06	
07	07	
08	08	
09	09	
10	19	
11	18	
12	17	
13	16	
14	15	
15	14	
16	13	
17	12	
18	11	
19	10	

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary
00	00	00
01	01	01
02	02	10
03	03	11
04	04	00
05	05	01
06	06	10
07	07	11
08	08	
09	09	
10	19	
11	18	
12	17	
13	16	
14	15	
15	14	
16	13	
17	12	
18	11	
19	10	

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary
00	00	000
01	01	001
02	02	010
03	03	011
04	04	100
05	05	101
06	06	110
07	07	111
08	08	
09	09	
10	19	
11	18	
12	17	
13	16	
14	15	
15	14	
16	13	
17	12	
18	11	
19	10	

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary
00	00	000
01	01	001
02	02	010
03	03	011
04	04	100
05	05	101
06	06	110
07	07	111
08	08	000
09	09	001
10	19	010
11	18	011
12	17	100
13	16	101
14	15	110
15	14	111
16	13	
17	12	
18	11	
19	10	

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary
00	00	0000
01	01	0001
02	02	0010
03	03	0011
04	04	0100
05	05	0101
06	06	0110
07	07	0111
08	08	1000
09	09	1001
10	19	1010
11	18	1011
12	17	1100
13	16	1101
14	15	1110
15	14	1111
16	13	
17	12	
18	11	
19	10	

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary	
00	00	0000	0
01	01	0001	1
02	02	0010	
03	03	0011	
04	04	0100	
05	05	0101	
06	06	0110	
07	07	0111	
08	08	1000	
09	09	1001	
10	19	1010	
11	18	1011	
12	17	1100	
13	16	1101	
14	15	1110	
15	14	1111	
16	13		
17	12		
18	11		
19	10		

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary	
00	00	0000	0
01	01	0001	1
02	02	0010	1
03	03	0011	0
04	04	0100	
05	05	0101	
06	06	0110	
07	07	0111	
08	08	1000	
09	09	1001	
10	19	1010	
11	18	1011	
12	17	1100	
13	16	1101	
14	15	1110	
15	14	1111	
16	13		
17	12		
18	11		
19	10		

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Decimal	RDC	Binary	
00	00	0000	00
01	01	0001	01
02	02	0010	11
03	03	0011	10
04	04	0100	
05	05	0101	
06	06	0110	
07	07	0111	
08	08	1000	
09	09	1001	
10	19	1010	
11	18	1011	
12	17	1100	
13	16	1101	
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16	13		
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Decimal	RDC	Binary	
00	00	0000	00
01	01	0001	01
02	02	0010	11
03	03	0011	10
04	04	0100	10
05	05	0101	11
06	06	0110	01
07	07	0111	00
08	08	1000	
09	09	1001	
10	19	1010	
11	18	1011	
12	17	1100	
13	16	1101	
14	15	1110	
15	14	1111	
16	13		
17	12		
18	11		
19	10		

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00	00	0000	000
01	01	0001	001
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04	04	0100	110
05	05	0101	111
06	06	0110	101
07	07	0111	100
08	08	1000	
09	09	1001	
10	19	1010	
11	18	1011	
12	17	1100	
13	16	1101	
14	15	1110	
15	14	1111	
16	13		
17	12		
18	11		
19	10		

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00	00	0000	000
01	01	0001	001
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04	04	0100	110
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06	06	0110	101
07	07	0111	100
08	08	1000	100
09	09	1001	101
10	19	1010	111
11	18	1011	110
12	17	1100	010
13	16	1101	011
14	15	1110	001
15	14	1111	000
16	13		
17	12		
18	11		
19	10		

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary	
00	00	0000	0000
01	01	0001	0001
02	02	0010	0011
03	03	0011	0010
04	04	0100	0110
05	05	0101	0111
06	06	0110	0101
07	07	0111	0100
08	08	1000	1100
09	09	1001	1101
10	19	1010	1111
11	18	1011	1110
12	17	1100	1010
13	16	1101	1011
14	15	1110	1001
15	14	1111	1000
16	13		
17	12		
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19	10		

Frank Gray's solution - Reflected Binary Counting

Decimal	RDC	Binary	RBC
00	00	0000	0000
01	01	0001	0001
02	02	0010	0011
03	03	0011	0010
04	04	0100	0110
05	05	0101	0111
06	06	0110	0101
07	07	0111	0100
08	08	1000	1100
09	09	1001	1101
10	19	1010	1111
11	18	1011	1110
12	17	1100	1010
13	16	1101	1011
14	15	1110	1001
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17	12		
18	11		
19	10		

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01	01	0001	0001
02	02	0010	0011
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04	04	0100	0110
05	05	0101	0111
06	06	0110	0101
07	07	0111	0100
08	08	1000	1100
09	09	1001	1101
10	19	1010	1111
11	18	1011	1110
12	17	1100	1010
13	16	1101	1011
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16	13		
17	12		
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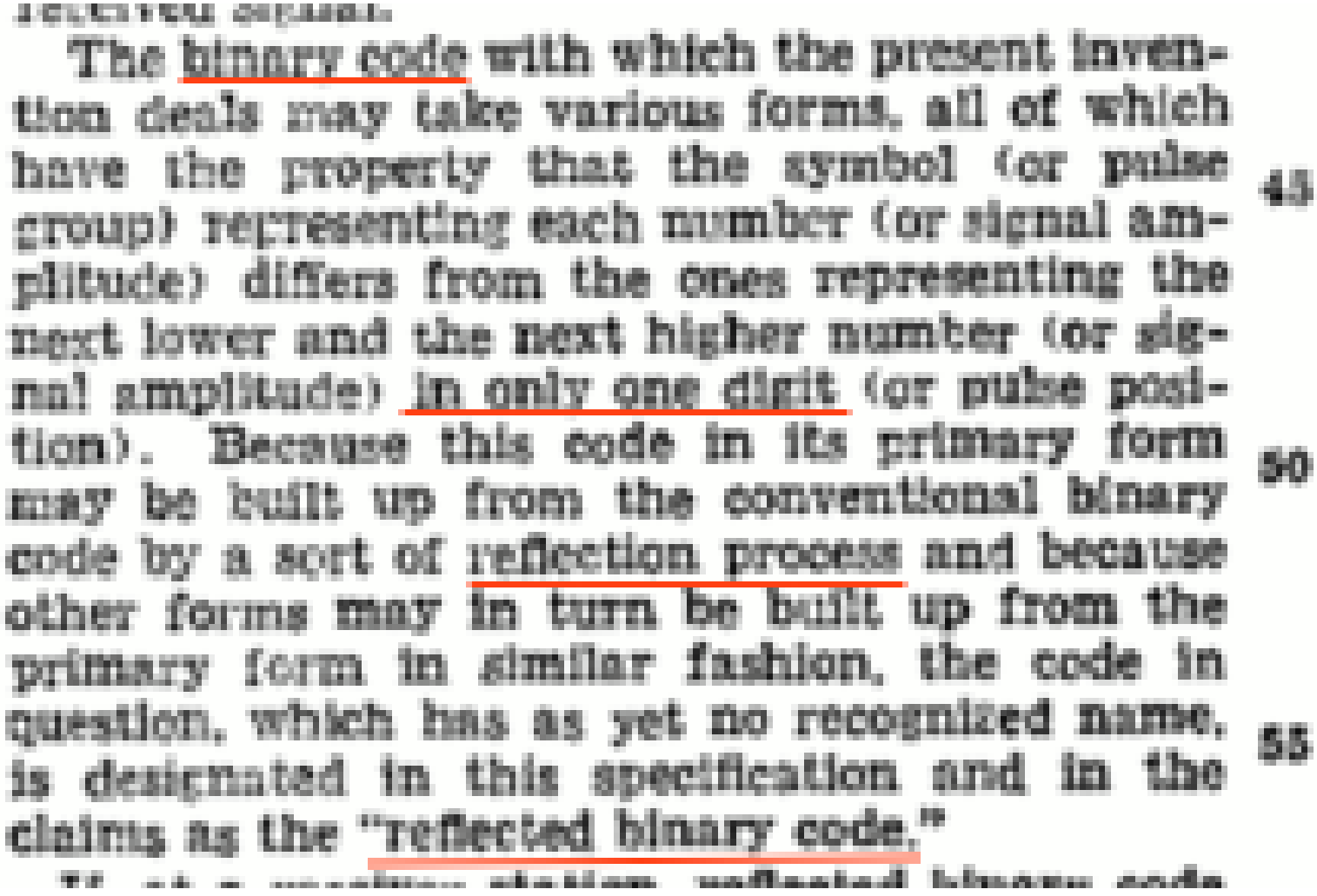
19	10		

Frank Gray

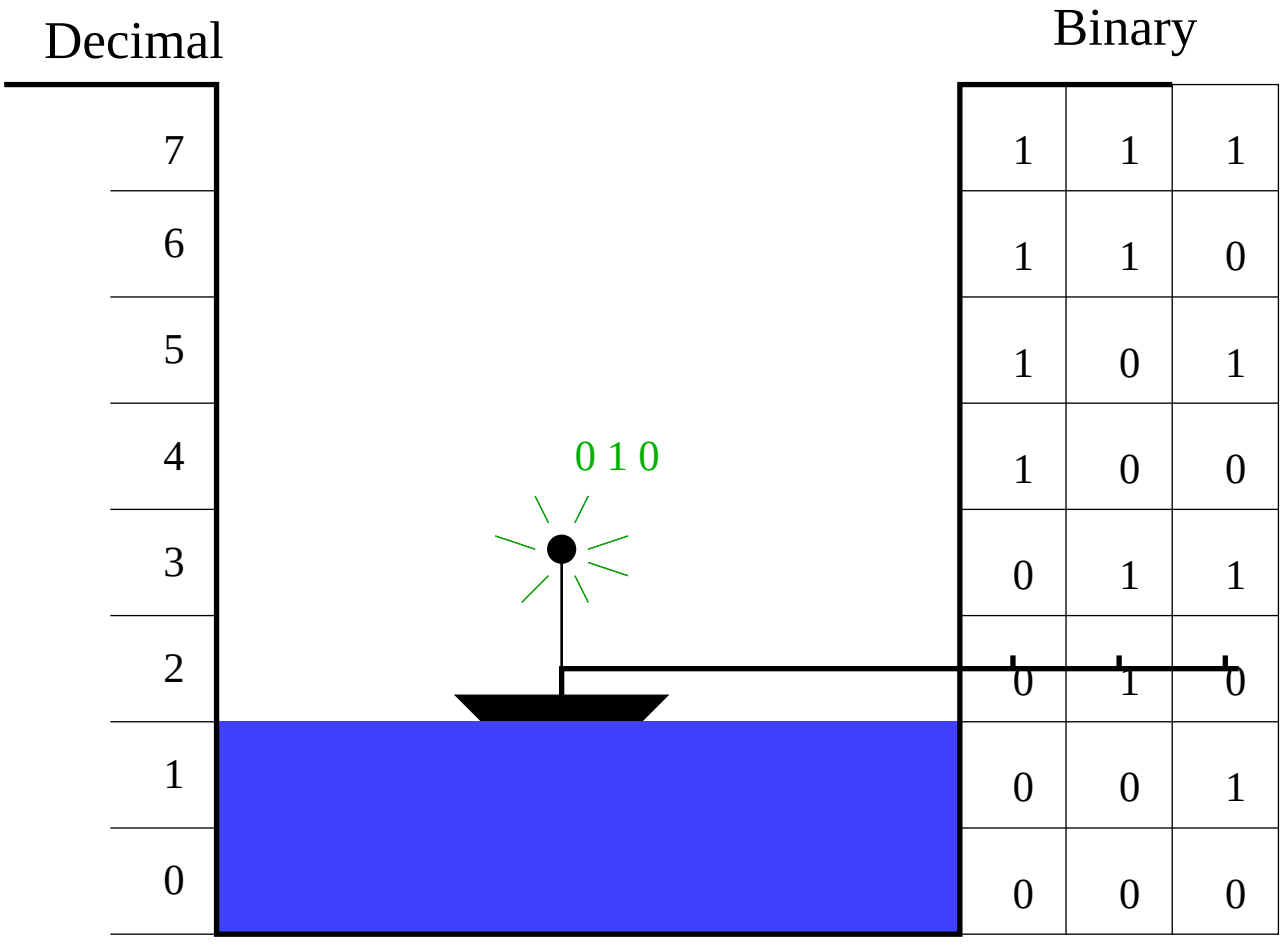
Frank Gray's solution - Reflected Binary Counting



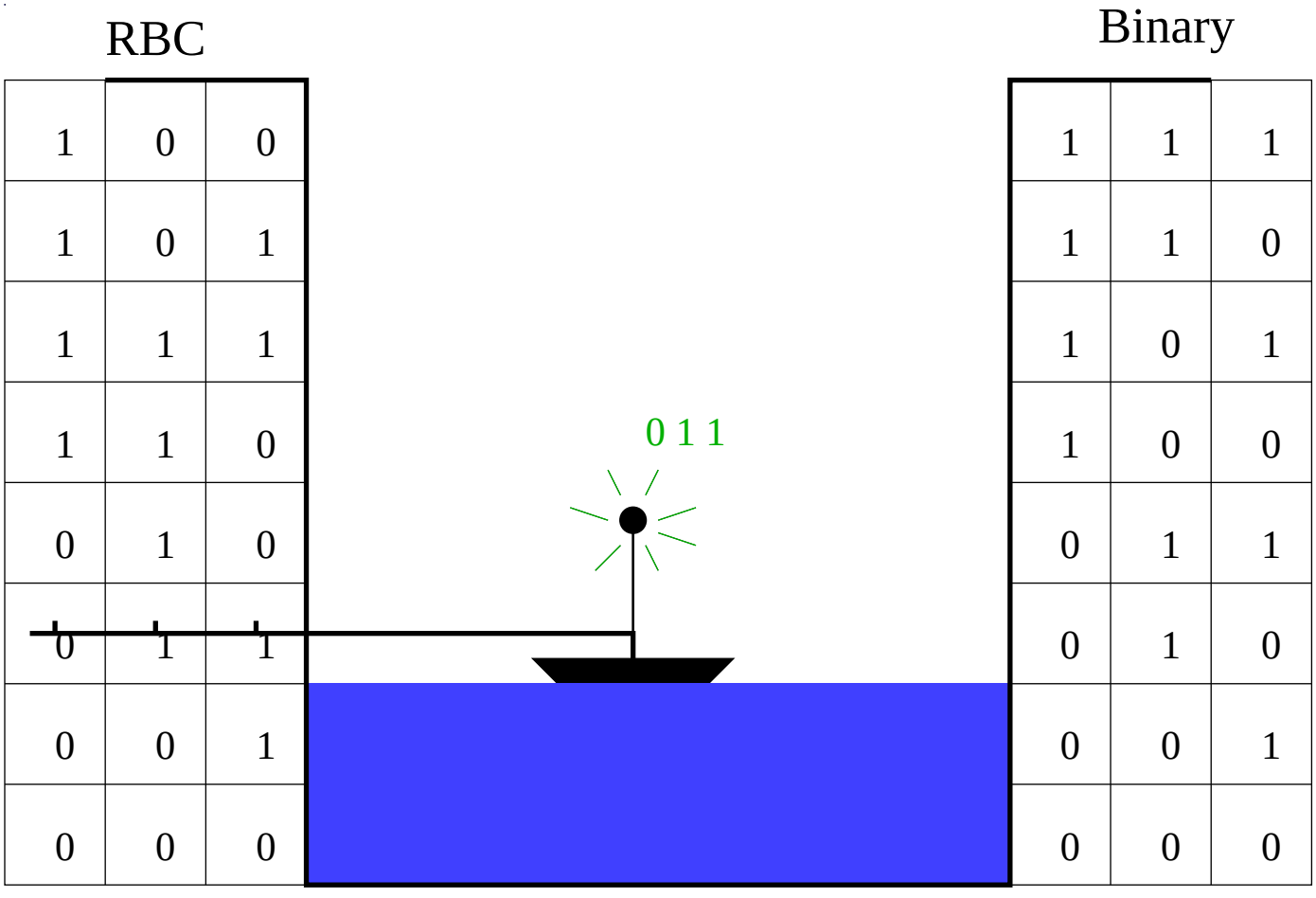
The binary code with which the present invention deals may take various forms, all of which have the property that the symbol (or pulse group) representing each number (or signal amplitude) differs from the ones representing the next lower and the next higher number (or signal amplitude) in only one digit (or pulse position). Because this code in its primary form may be built up from the conventional binary code by a sort of reflection process and because other forms may in turn be built up from the primary form in similar fashion, the code in question, which has as yet no recognized name, is designated in this specification and in the claims as the "reflected binary code."

Frank Gray's Patent Application 1947

Frank Gray's solution - Reflected Binary Counting



Frank Gray's solution - Reflected Binary Counting



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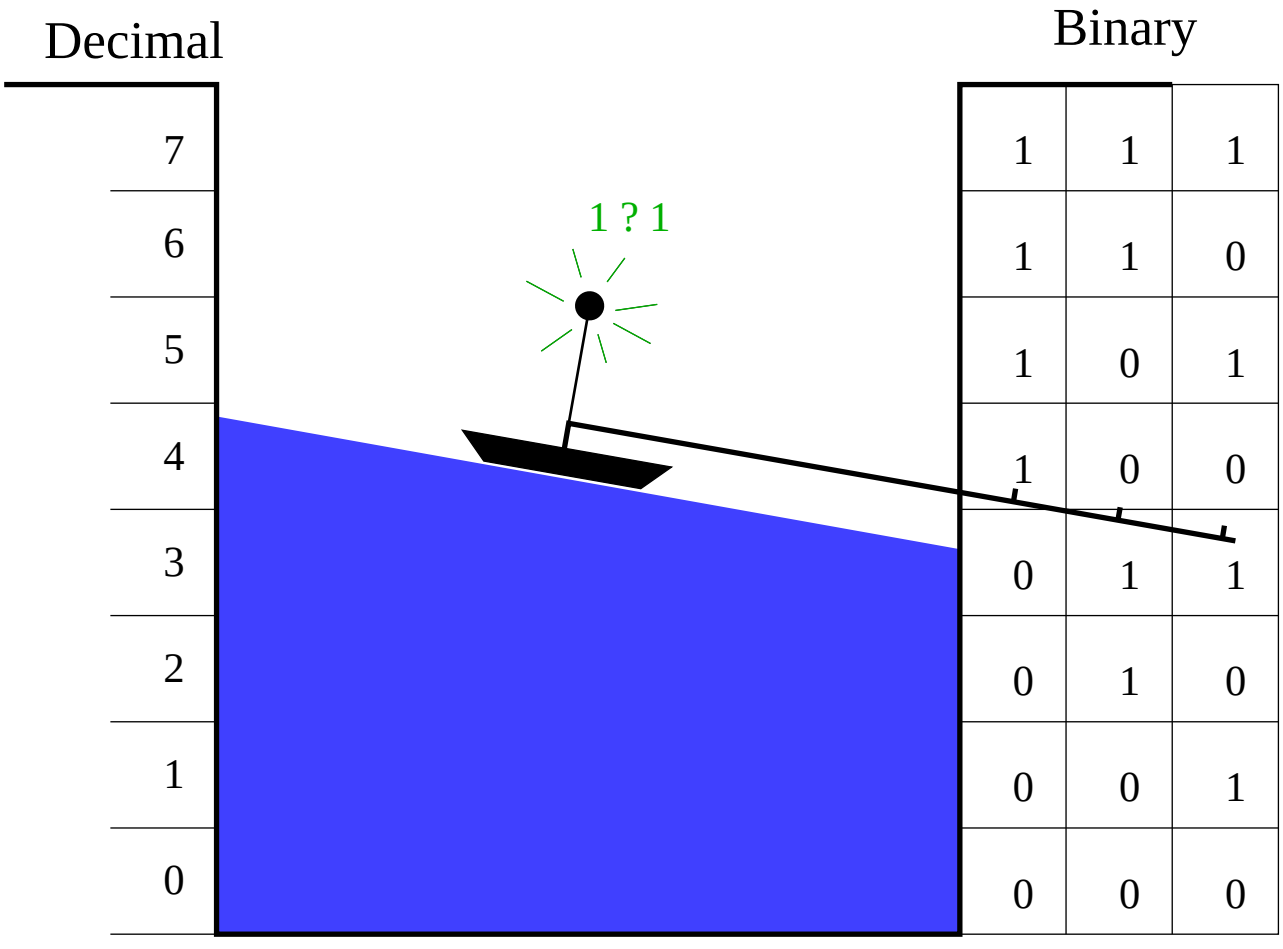
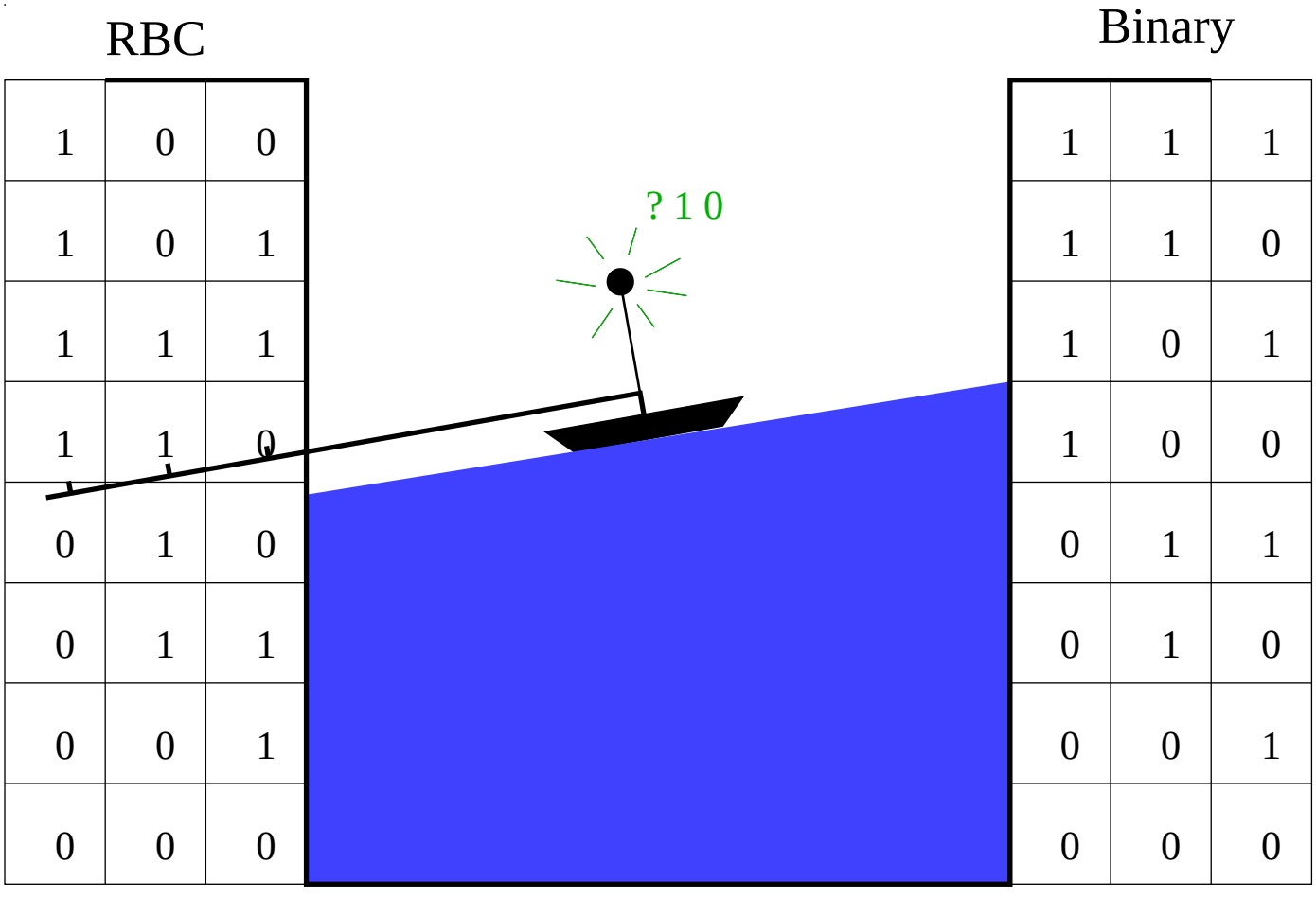


Binary counting gives large position errors

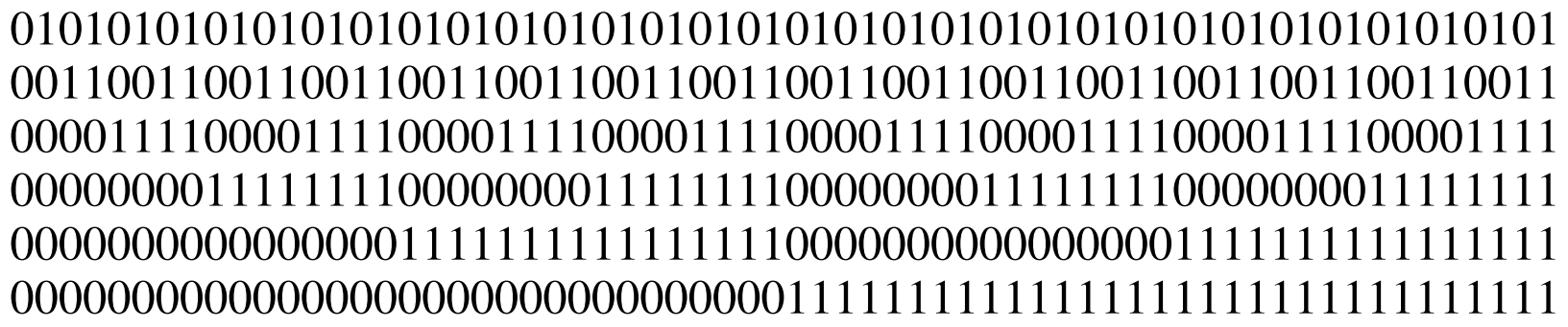
Frank Gray's solution - Reflected Binary Counting



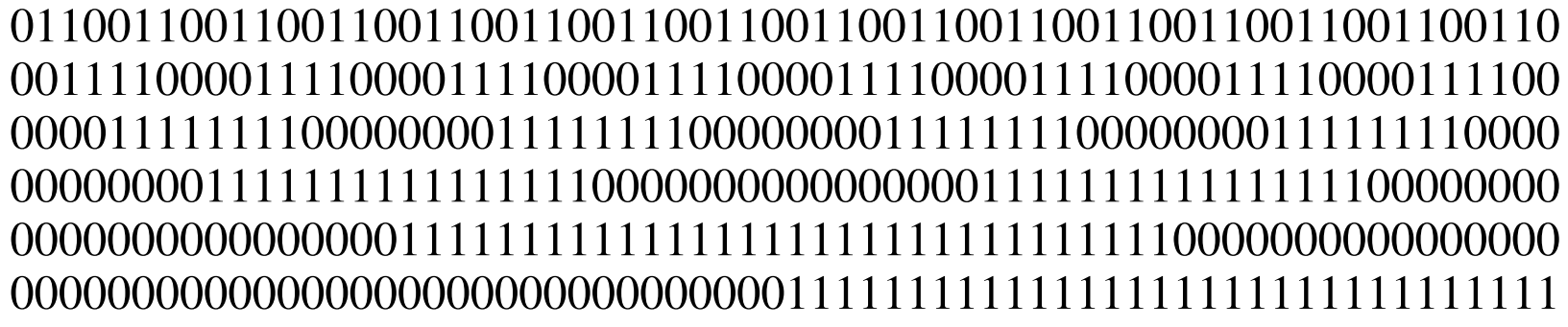
Advantage: RBC eliminates large position errors

Frank Gray's solution - Reflected Binary Counting

Binary Counting



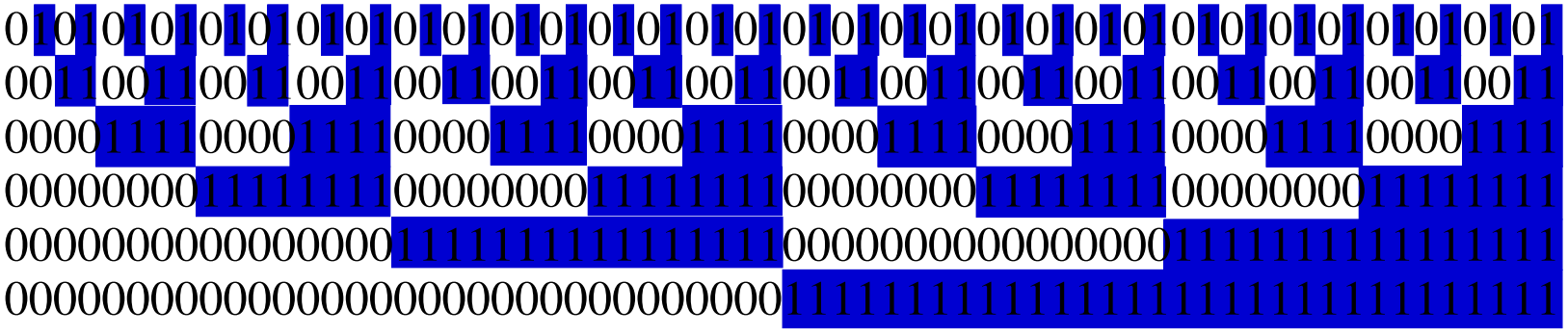
RBC Counting



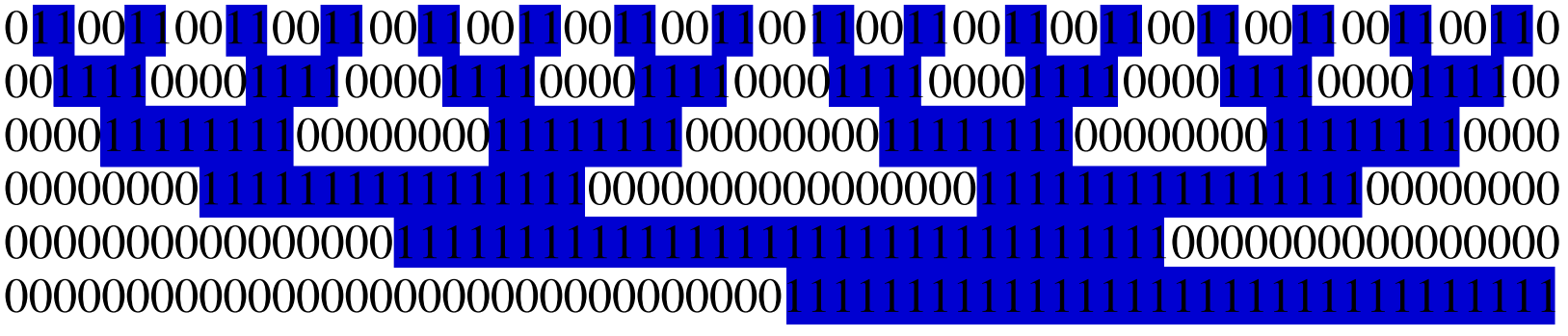
Fractal Mask Shapes

Frank Gray's solution - Reflected Binary Counting

Binary Counting



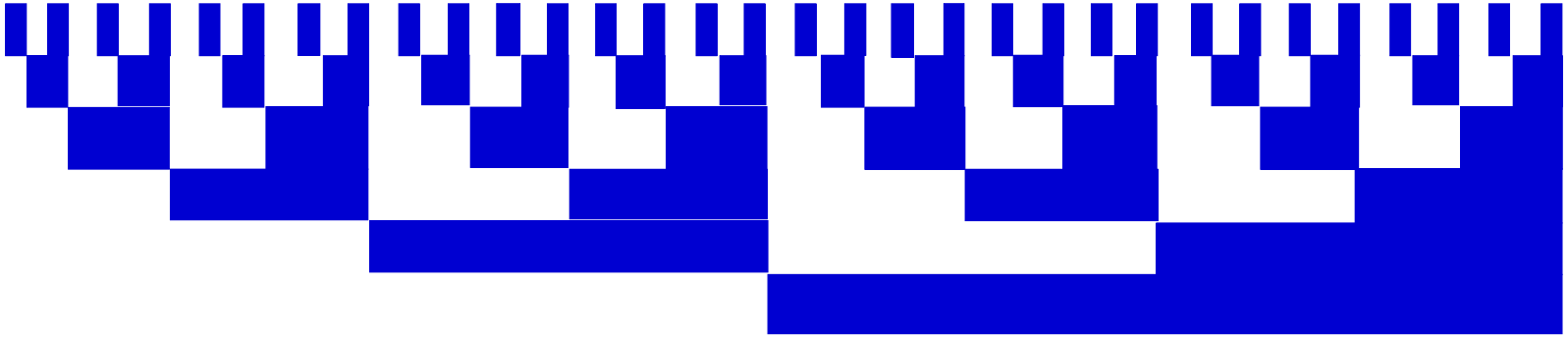
RBC Counting



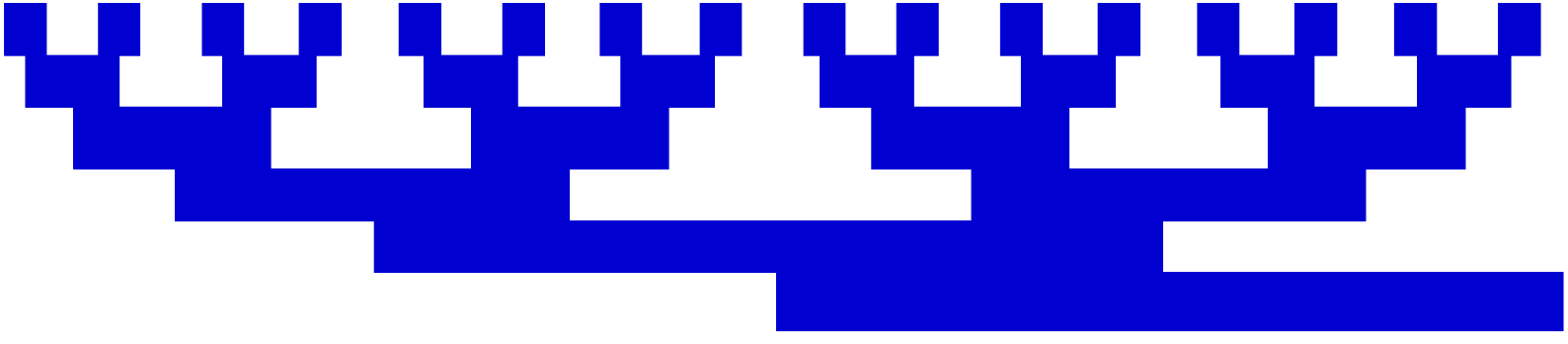
Fractal Mask Shapes

Frank Gray's solution - Reflected Binary Counting

Binary Counting



RBC Counting



Fractal Mask Shapes

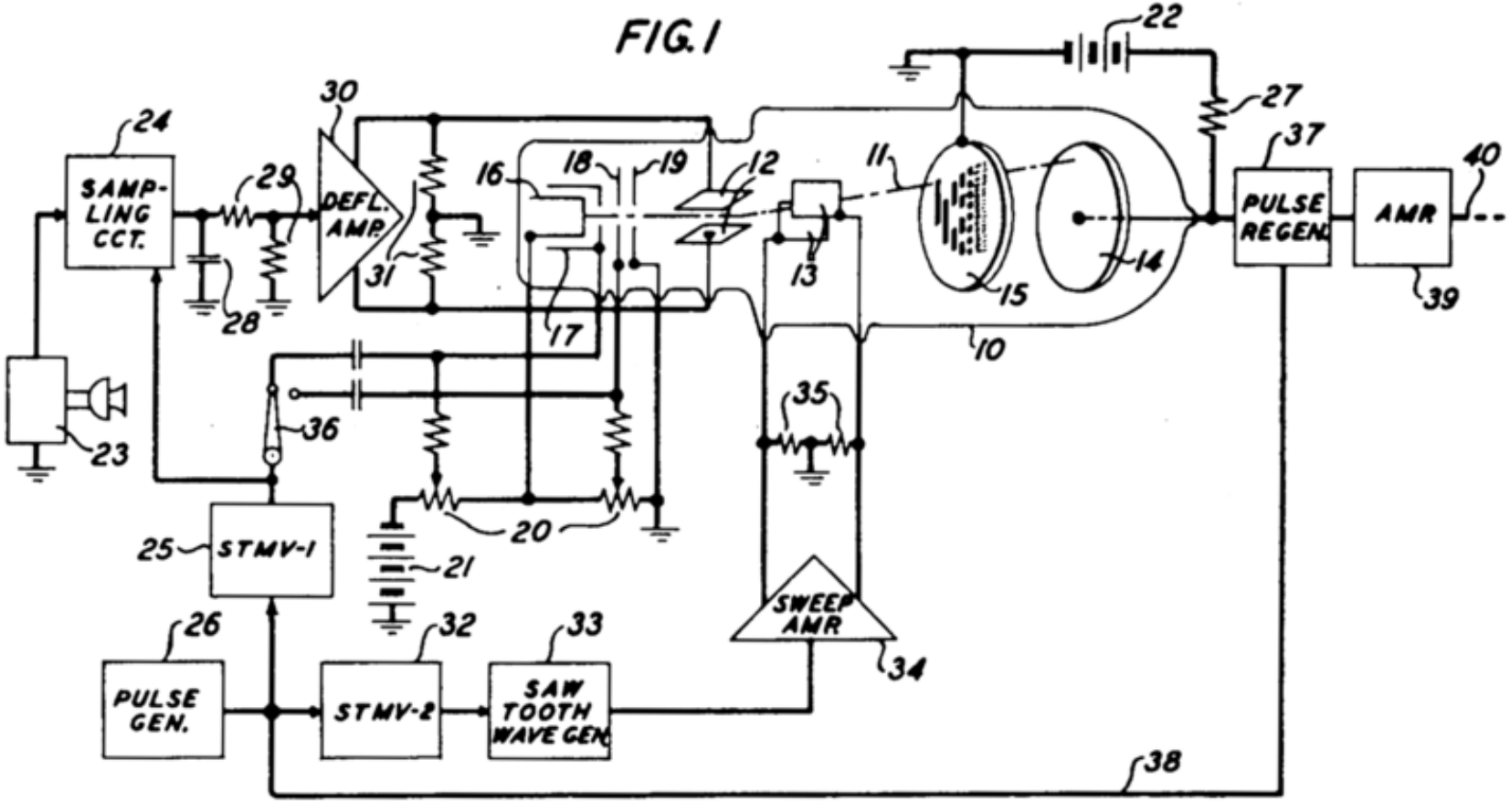
Frank Gray's solution - Reflected Binary Counting

March 17, 1953 F. GRAY 2,632,058

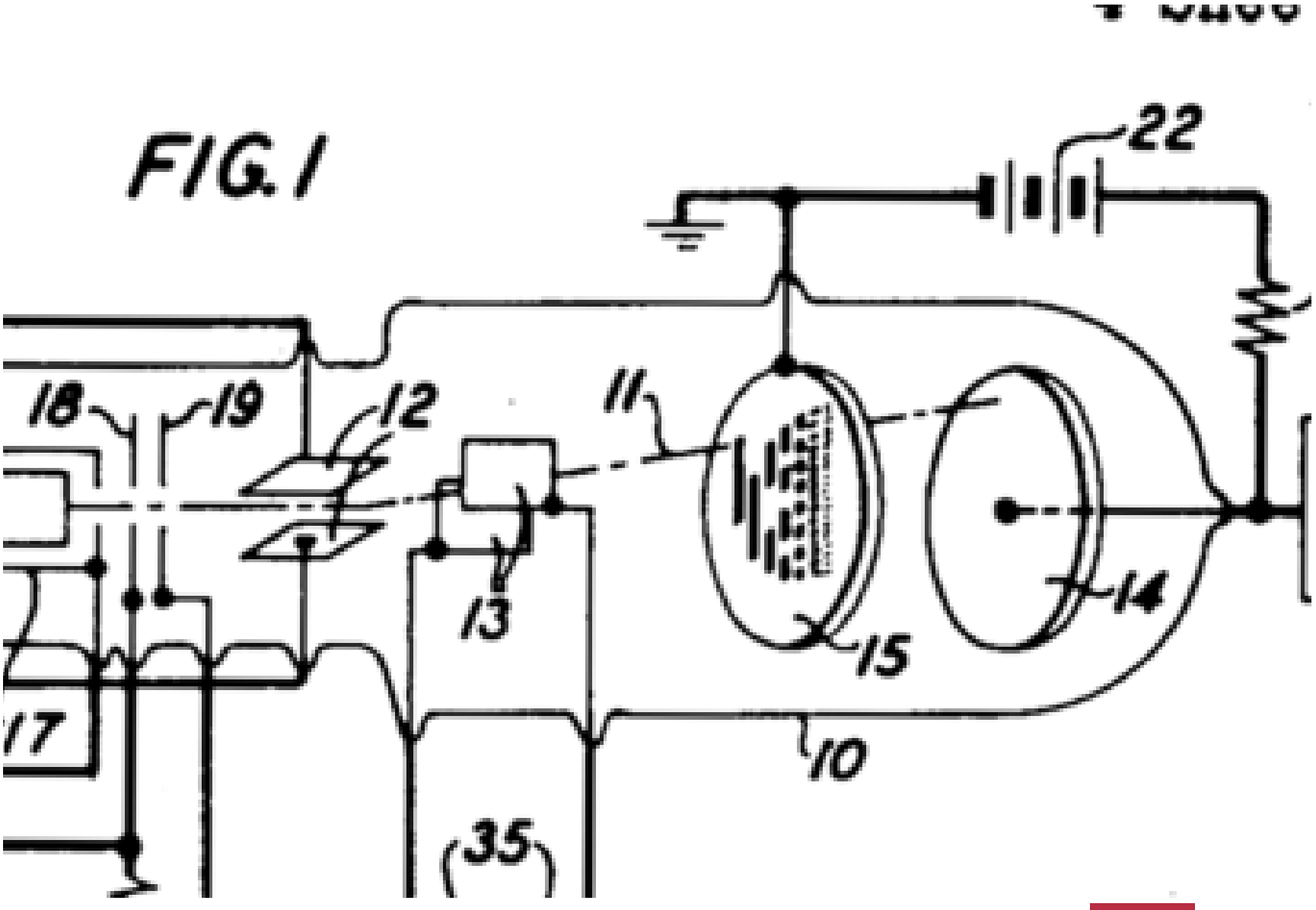
PULSE CODE COMMUNICATION

Filed Nov. 13, 1947 4 Sheets-Sheet 1

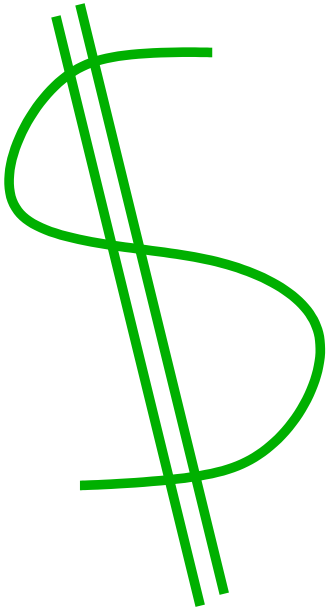
FIG. 1

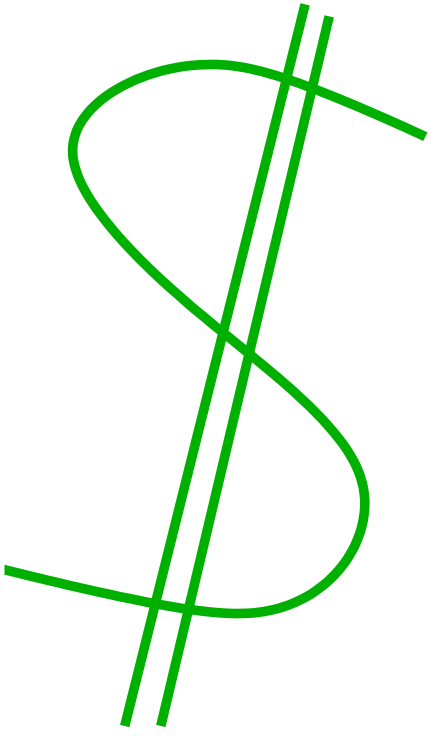


Frank Gray's solution - Reflected Binary Counting



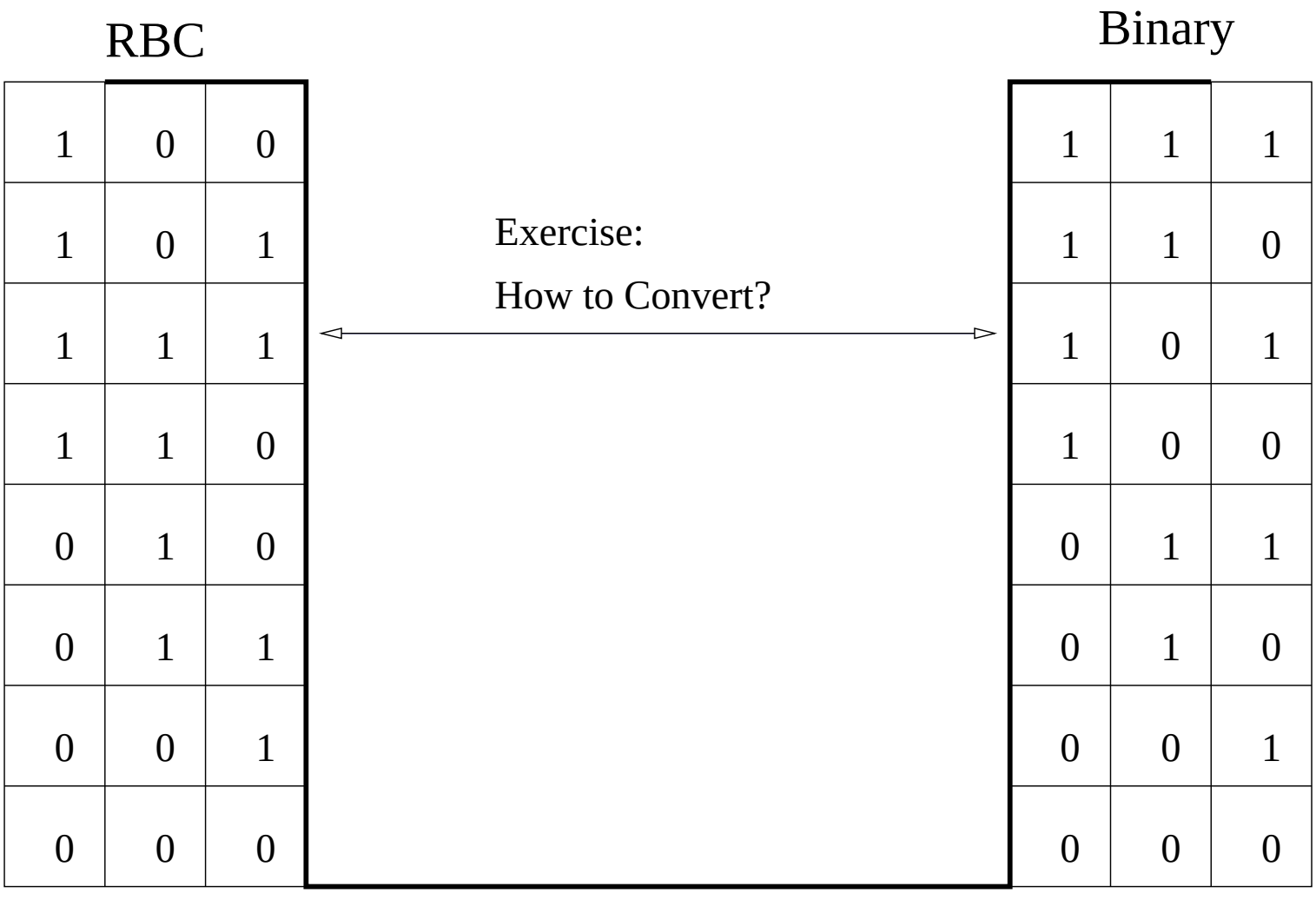
Frank Gray's solution - Reflected Binary Counting





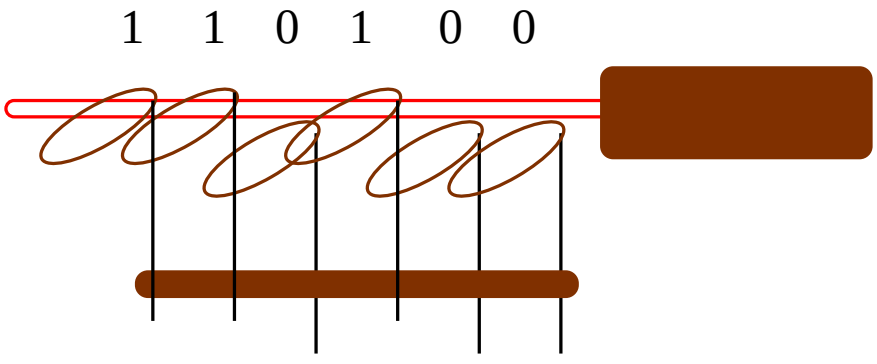
Patent 2,632,058 earnings:
About $1,000,000,00

Frank Gray's solution - Reflected Binary Counting

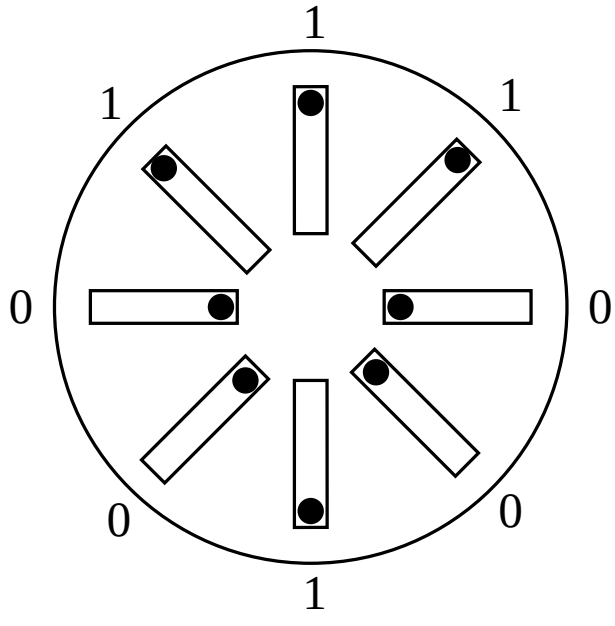


RBC			Binary		
1	0	0	1	1	1
1	0	1	1	1	0
1	1	1	1	0	1
1	1	0	1	0	0
0	1	0	0	1	1
0	1	1	0	1	0
0	0	1	0	0	1
0	0	0	0	0	0

RBC in puzzles

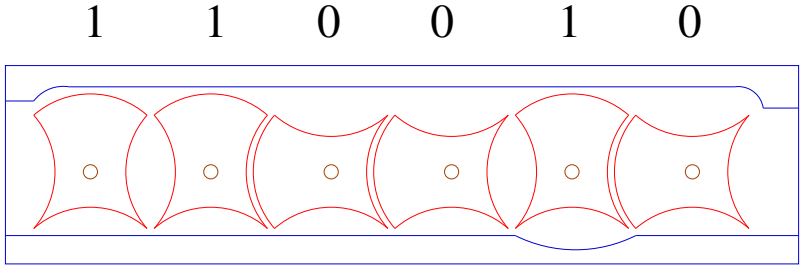


Chinese Rings

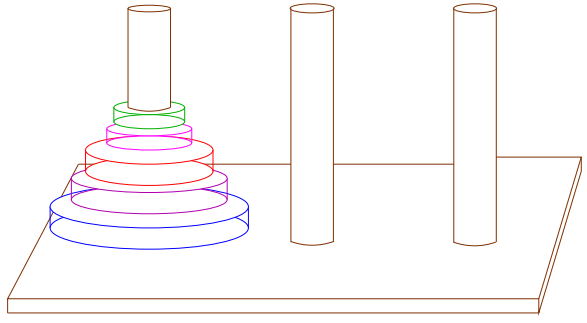


The Brain

RBC in puzzles



Spin Out!



Tower of Hanoi

Specialization 1: Time trend resistance

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

LexOrder

Time ↓

H	T	N	P
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

LexOrder

Time

H	T	N	P
0	0	0	0
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1
0	1	1	0
0	1	1	1
1	0	0	0
1	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
1	1	0	1
1	1	1	0
1	1	1	1

26 changes

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

Time ↓

LexOrder				RBC (Fewer Changes)			
H	T	N	P	H	T	N	P
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	1
0	0	1	1	0	0	1	0
0	1	0	0	0	1	1	0
0	1	0	1	0	1	1	1
0	1	1	0	0	1	0	1
0	1	1	1	0	1	0	0
1	0	0	0	1	1	0	0
1	0	0	1	1	1	0	1
1	0	1	0	1	1	1	1
1	0	1	1	1	1	1	0
1	1	0	0	1	0	1	0
1	1	0	1	1	0	1	1
1	1	1	0	1	0	0	1
1	1	1	1	1	0	0	0
26 changes				15 changes			

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

RBC
(Fewer Changes)

H	T	N	P
0	0	0	0
0	0	0	1
0	0	1	1
0	0	1	0
0	1	1	0
0	1	1	1
0	1	0	1
0	1	0	0
1	1	0	0
1	1	0	1
1	1	1	1
1	1	1	0
1	0	1	0
1	0	1	1
1	0	0	1
1	0	0	0

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

RBC
(Fewer Changes)

H	T	N	P	
0	0	0	0	
0	0	0	1	
0	0	1	1	
0	0	1	0	
0	1	1	0	
0	1	1	1	
0	1	0	1	
0	1	0	0	
1	1	0	0	
1	1	0	1	
1	1	1	1	
1	1	1	0	
1	0	1	0	
1	0	1	1	Best
1	0	0	1	
1	0	0	0	

Time

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

RBC
(Fewer Changes)

H	T	N	P	
0	0	0	0	
0	0	0	1	
0	0	1	1	
0	0	1	0	
0	1	1	0	
0	1	1	1	
0	1	0	1	
0	1	0	0	
1	1	0	0	
1	1	0	1	
1	1	1	1	
1	1	1	0	
1	0	1	0	
1	0	1	1	Best?
1	0	0	1	
1	0	0	0	

Time

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

RBC
(Fewer Changes)

H T N P

0 0 0 0
0 0 0 1
0 0 1 1 Best?
0 0 1 0
0 1 1 0
0 1 1 1
0 1 0 1
0 1 0 0
1 1 0 0
1 1 0 1
1 1 1 1
1 1 1 0
1 0 1 0
1 0 1 1 Best?
1 0 0 1
1 0 0 0

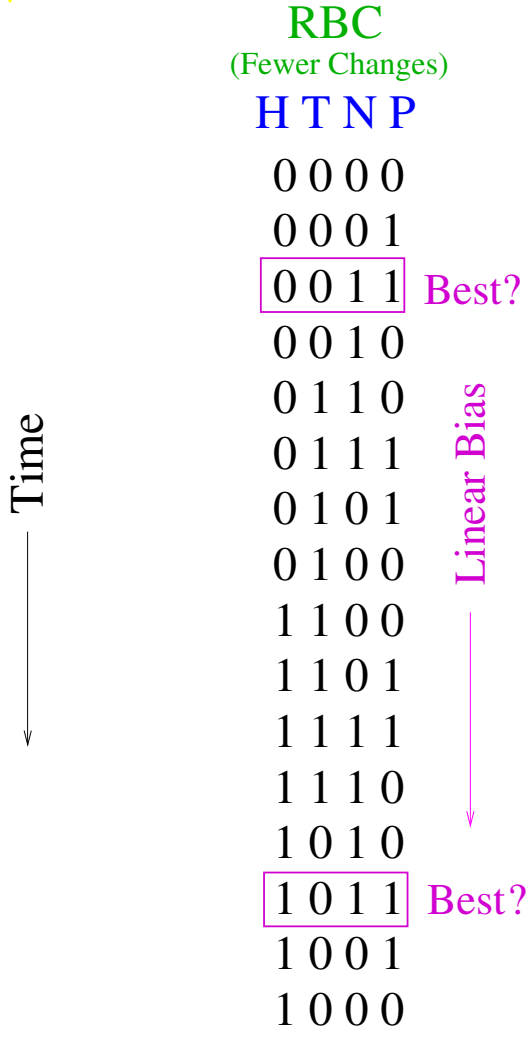
Time

Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

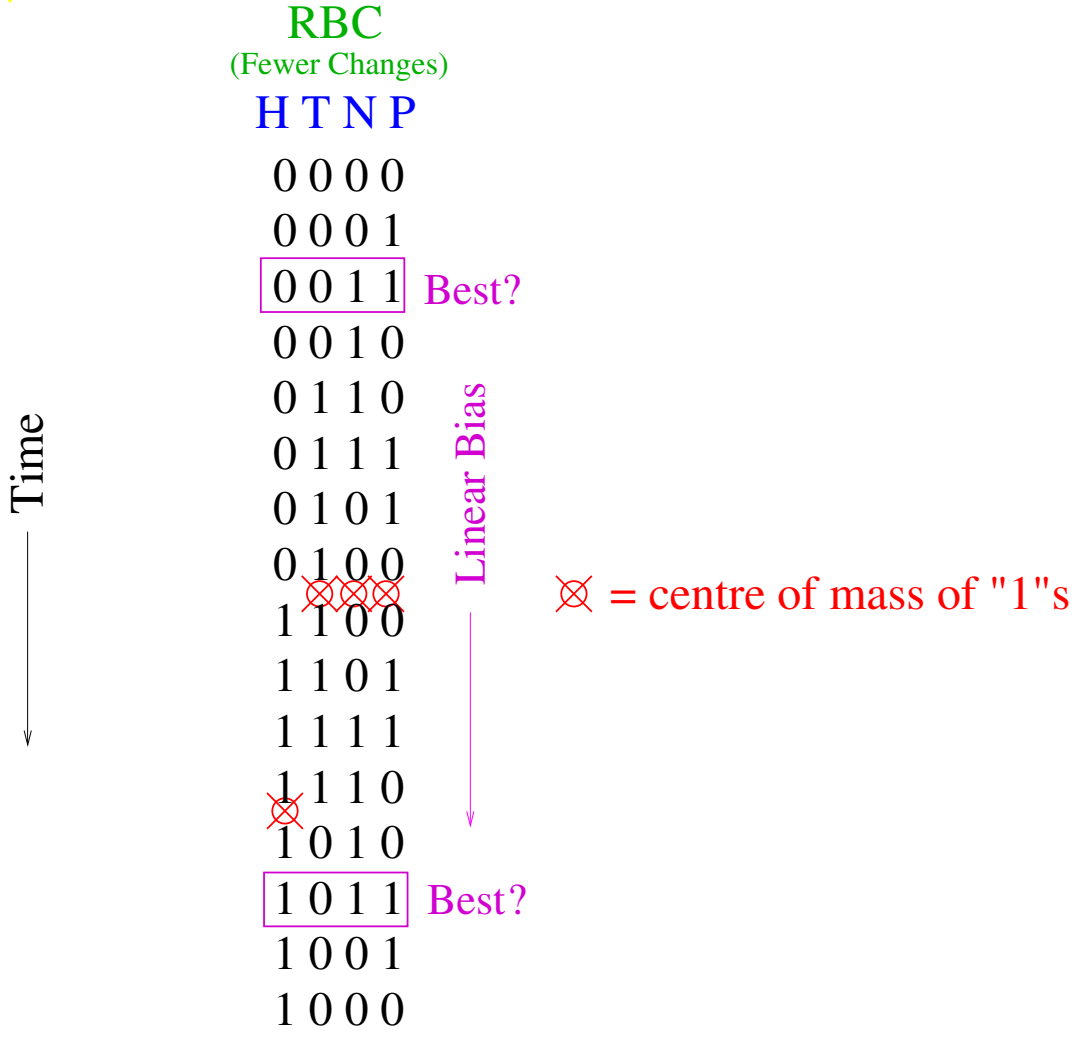


Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

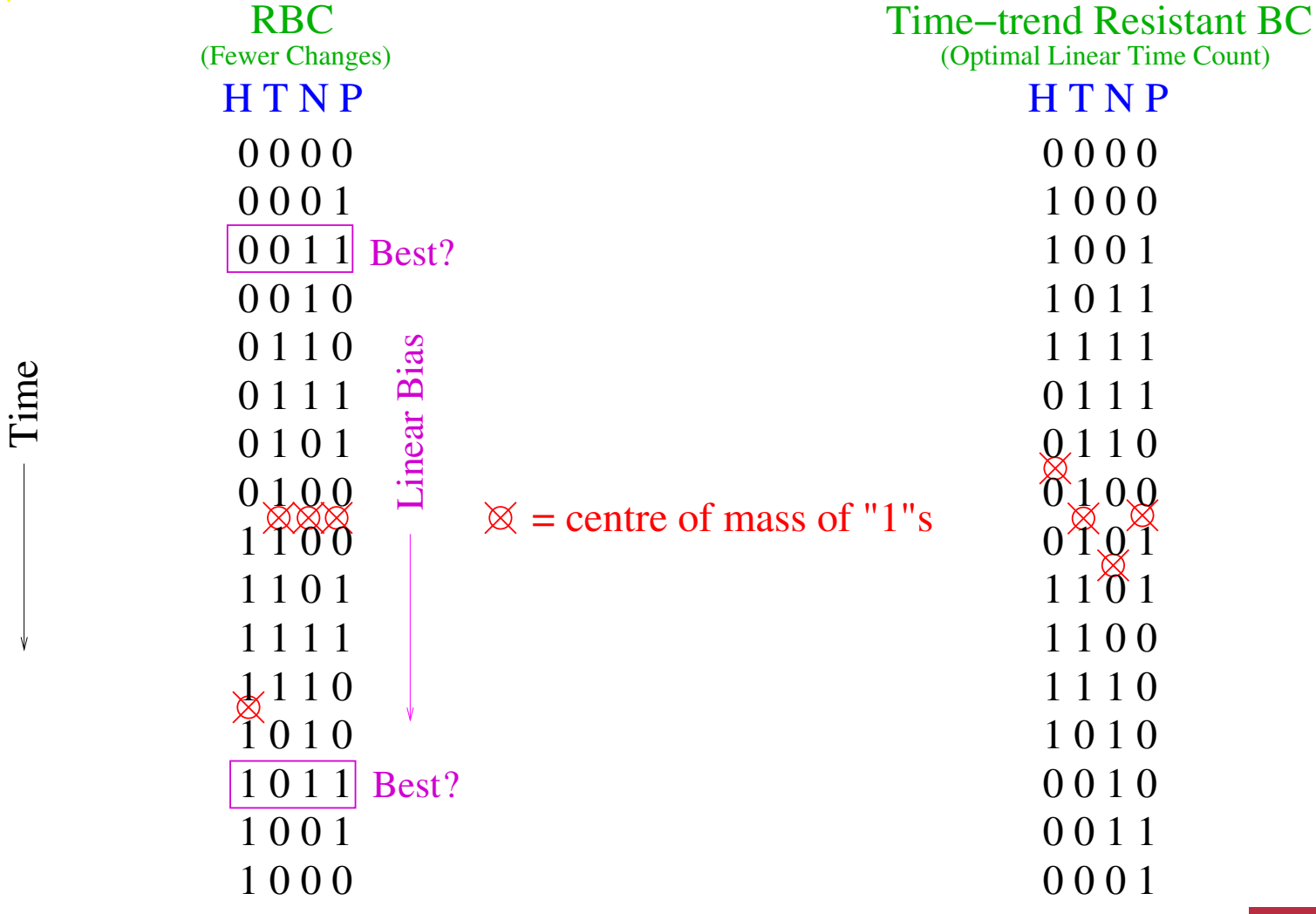


Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high

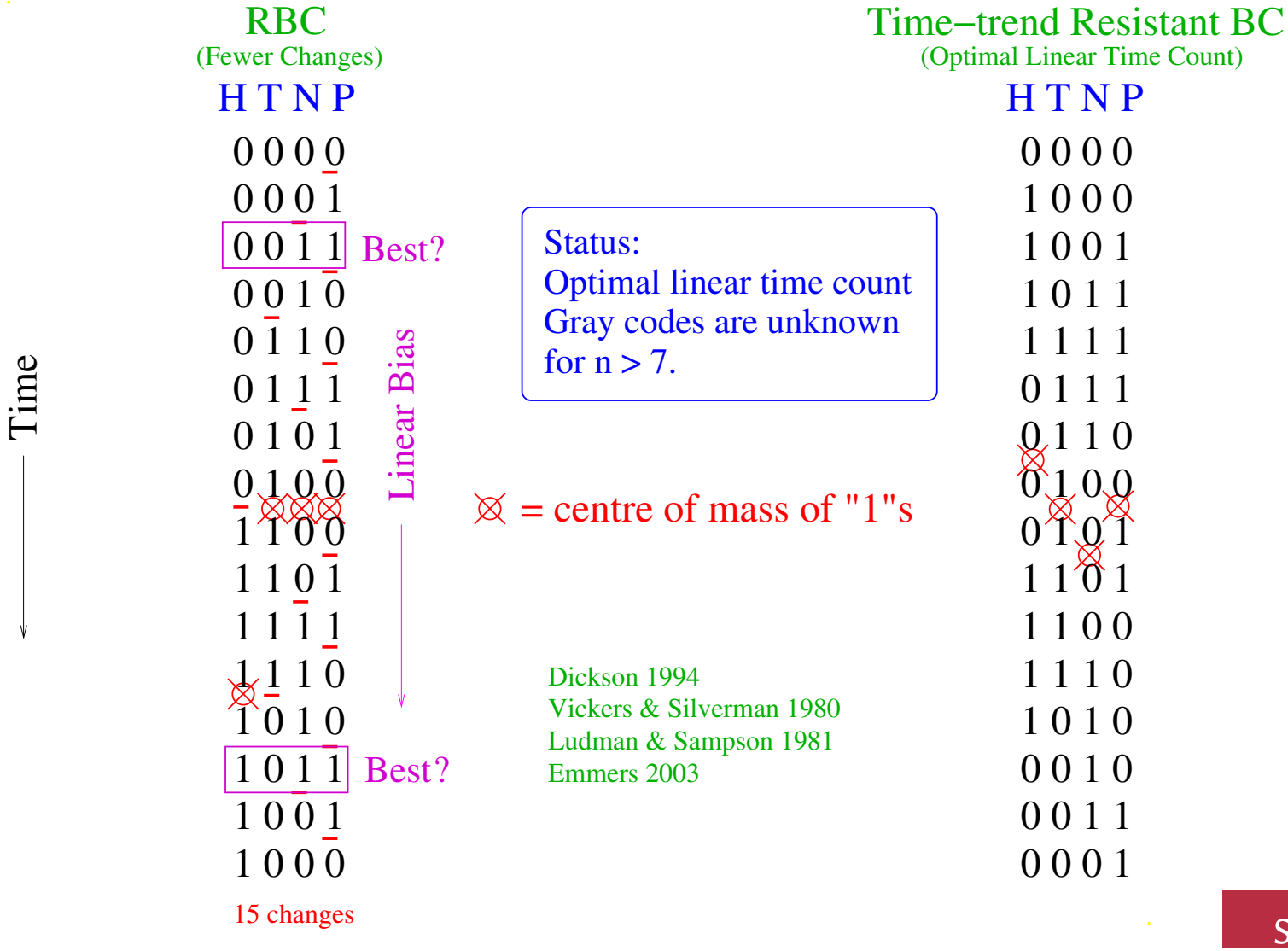


Specialization 1: Time trend resistance

4-factor Plant Growth Experiment

Factors: Humidity Temprature Nitrogen Phospheros

Levels: 0 = low 1 = high



Status:
Optimal linear time count
Gray codes are unknown
for n > 7.

Dickson 1994
Vickers & Silverman 1980
Ludman & Sampson 1981
Emmers 2003

Specialization 2: Balanced transition count

Specialization 2: Balanced transition count

RBC

0 0 0 0
0 0 0 1
0 0 1 1
0 0 1 0
0 1 1 0
0 1 1 1
0 1 0 1
0 1 0 0
1 1 0 0
1 1 0 1
1 1 1 1
1 1 1 0
1 0 1 0
1 0 1 1
1 0 0 1
1 0 0 0

Specialization 2: Balanced transition count

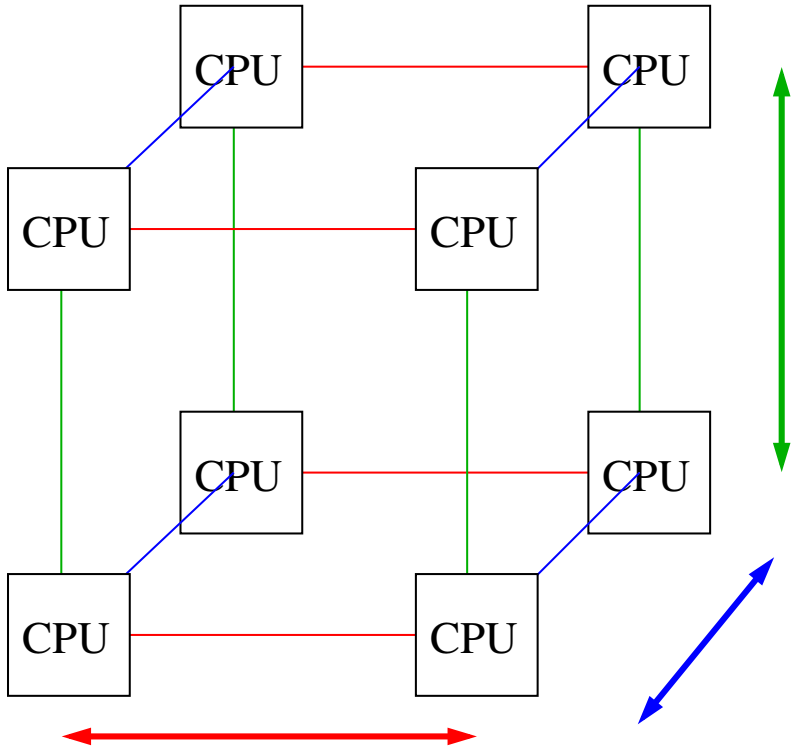
RBC
0 0 0 0
0 0 0 1
0 0 1 1
0 0 1 0
0 1 1 0
0 1 1 1
0 1 0 1
0 1 0 0
1 1 0 0
1 1 0 1
1 1 1 1
1 1 1 0
1 0 1 0
1 0 1 1
1 0 0 1
1 0 0 0
2 2 4 8

Specialization 2: Balanced transition count

RBC	Transition Balanced GC
0 0 0 0	0 0 0 0
0 0 0 1	0 1 0 0
0 0 1 1	0 1 1 0
0 0 1 0	1 1 1 0
0 1 1 0	1 0 1 0
0 1 1 1	0 0 1 0
0 1 0 1	0 0 1 1
0 1 0 0	1 0 1 1
1 1 0 0	1 0 0 1
1 1 0 1	1 0 0 0
1 1 1 1	1 1 0 0
1 1 1 0	1 1 0 1
1 0 1 0	1 1 1 1
1 0 1 1	0 1 1 1
1 0 0 1	0 1 0 1
1 0 0 0	0 0 0 1
2 2 4 8	4 4 4 4

Specialization 2: Balanced transition count

RBC	Transition Balanced GC
0 0 0 0	0 0 0 0
0 0 0 1	0 1 0 0
0 0 1 1	0 1 1 0
0 0 1 0	1 1 1 0
0 1 1 0	1 0 1 0
0 1 1 1	0 0 1 0
0 1 0 1	0 0 1 1
0 1 0 0	1 0 1 1
1 1 0 0	1 0 0 1
1 1 0 1	1 0 0 0
1 1 1 1	1 1 0 0
1 1 1 0	1 1 0 1
1 0 1 0	1 1 1 1
1 0 1 1	0 1 1 1
1 0 0 1	0 1 0 1
1 0 0 0	0 0 0 1
2 2 4 8	4 4 4 4



Load balancing for hypercube architecture
(Woo 1991, Flahive 2006)

Specialization 2: Balanced transition count

RBC	Transition Balanced GC
0 0 0 0	0 0 0 0
0 0 0 1	0 1 0 0
0 0 1 1	0 1 1 0
0 0 1 0	1 1 1 0
0 1 1 0	1 0 1 0
0 1 1 1	0 0 1 0
0 1 0 1	0 0 1 1
0 1 0 0	1 0 1 1
1 1 0 0	1 0 0 1
1 1 0 1	1 0 0 0
1 1 1 1	1 1 0 0
1 1 1 0	1 1 0 1
1 0 1 0	1 1 1 1
1 0 1 1	0 1 1 1
1 0 0 1	0 1 0 1
1 0 0 0	0 0 0 1
2 2 4 8	4 4 4 4

Theorem

For every n, there exists an n–bit Gray code which is as balanced as possible.

$$|t_i - t_j| < 2$$

Proof

Technical construction

$$n \longrightarrow n+2$$

Specialization 2: Balanced transition count

RBC	Transition Balanced GC
0 0 0 0	0 0 0 0
0 0 0 1	0 1 0 0
0 0 1 1	0 1 1 0
0 0 1 0	1 1 1 0
0 1 1 0	1 0 1 0
0 1 1 1	0 0 1 0
0 1 0 1	0 0 1 1
0 1 0 0	1 0 1 1
1 1 0 0	1 0 0 1
1 1 0 1	1 0 0 0
1 1 1 1	1 1 0 0
1 1 1 0	1 1 0 1
1 0 1 0	1 1 1 1
1 0 1 1	0 1 1 1
1 0 0 1	0 1 0 1
1 0 0 0	0 0 0 1
2 2 4 8	4 4 4 4

Robinson & Cohn 1981
Wagner & West 1991
Bhat & Savage 1996
T. Bakos (via A. Adam) 1968
Supara & Jan & Zanten 2006
M. Flahive 2008

Theorem

For every n, there exists an n–bit Gray code which is as balanced as possible.

$$|t_i - t_j| < 2$$

Proof

Technical construction

$$n \longrightarrow n+2$$

Specialization 3: Sam Beckett's "Quad" problem

Specialization 3: Sam Beckett's "Quad" problem

Dramatic works [edit]

Theatre

- *Eleutheria* (1940s; published 1995)
- *Waiting for Godot* (1952)
- *Act Without Words I* (1956)
- *Act Without Words II* (1956)
- *Endgame* (1957)
- *Krapp's Last Tape* (1958)
- *Rough for Theatre I* (late 1950s)
- *Rough for Theatre II* (late 1950s)
- *Happy Days* (1960)
- *Play* (1963)
- *Come and Go* (1965)
- *Breath* (1969)
- *Not I* (1972)
- *That Time* (1975)
- *Footfalls* (1975)
- *A Piece of Monologue* (1980)
- *Rockaby* (1981)
- *Ohio Impromptu* (1981)
- *Catastrophe* (1982)
- *What Where* (1983)

Radio

- *All That Fall* (1956)
- *From an Abandoned Work* (1957)
- *Embers* (1959)
- *Rough for Radio I* (1961)
- *Rough for Radio II* (1961)
- *Words and Music* (1961)
- *Cascando* (1962)

Television

- *Eh Joe* (1965)
- *Ghost Trio* (1975)
- *... but the clouds ...* (1976)
- *Quad I + II* (1981)
- *Nacht und Träume* (1982)
- *Beckett Directs Beckett* (1988/92) The San Quentin Drama Workshop
- *Beckett on Film* (2002) Hosted by Jeremy Irons, Produced by PBS [39]

Cinema

- *Film* (1965)

Specialization 3: Sam Beckett's "Quad" problem

Poetry

- *Whoroscope* (1930)
- *Echo's Bones and other Precipitates* (1935)
- *Collected Poems in English* (1961)
- *Collected Poems in English and French* (1977)
- *What is the Word* (1989)
- *Selected Poems 1930-1989* (2009)

[edit]Translations

- *Anna Livia Plurabelle* (James Joyce, French translation by Bec and others) (1931)
- *Negro: an Anthology* (Nancy Cunard, editor) (1934)
- *Anthology of Mexican Poems* (Octavio Paz, editor) (1958)
- *The Old Tune* (Robert Pinget) (1963)
- *What Is Surrealism?: Selected Essays* (André Breton) (various pieces in the collection)

Specialization 3: Sam Beckett's "Quad" problem

Prose

Novels

- *Dream of Fair to Middling Women* (1932; published 1992)
- *Murphy* (1938)
- *Watt* (1945; published 1953)
- *Mercier and Camier* (1946; published 1974)
- *Molloy* (1951)
- *Malone Dies* (1951)
- *The Unnamable* (1953)
- *How It Is* (1961)

Novellas

- *The Expelled* (1946)
- *The Calmative* (1946)
- *The End* (1946)
- *The Lost Ones* (1971)
- *Company* (1980)
- *Ill Seen Ill Said* (1981)
- *Worstward Ho* (1983)

Stories

- *More Pricks Than Kicks* (1934)
- *First Love* (1945)
- *Stories and Texts for Nothing* (1954)
- *Fizzles* (1976)
- *Stirrings Still* (1988)

Non-fiction

- *Proust* (1931)
- *Three Dialogues* (with Georges Duthuit and Jacc
- *Disjecta* (1929 - 1967)
- *Dante...Bruno. Vico..Joyce*

Specialization 3: Sam Beckett's "Quad" problem

"Quad" Absurdist Theatre

Specialization 3: Sam Beckett's "Quad" problem

Specialization 3: Sam Beckett's "Quad" problem

Specialization 3: Sam Beckett's "Quad" problem

Sam's Rules:

1. The play begins and ends with an empty stage
2. One actor enters or leaves after each scene
3. Only the most tired actor can leave the stage
4. Every subset of actors appears exactly once

Movements and Stages

Stage	Series 1				Series 2				Series 3				Series 4			
One	white	-	-	-	yellow	-	-	-	blue	-	-	-	red	-	-	-
Two	white	blue	-	-	yellow	white	-	-	blue	yellow	-	-	red	blue	-	-
Three	white	blue	red	-	yellow	white	red	-	blue	yellow	white	-	red	blue	yellow	-
Four	white	blue	red	yellow	yellow	white	red	blue	blue	yellow	white	red	red	blue	yellow	white
Five	-	blue	red	yellow	-	white	red	blue	-	yellow	white	red	-	blue	yellow	white
Six	-	-	red	yellow	-	-	red	blue	-	-	white	red	-	-	yellow	white

Specialization 3: Sam Beckett's "Quad" problem

Sam's Rules:

1. The play begins and ends with an empty stage
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Movements and Stages

Stage	Series 1				Series 2				Series 3				Series 4			
One	white	-	-	-	yellow	-	-	-	blue	-	-	-	red	-	-	-
Two	white	blue	-	-	yellow	white	-	-	blue	yellow	-	-	red	blue	-	-
Three	white	blue	red	-	yellow	white	red	-	blue	yellow	white	-	red	blue	yellow	-
Four	white	blue	red	yellow	yellow	white	red	blue	blue	yellow	white	red	red	blue	yellow	white
Five	-	blue	red	yellow	-	white	red	blue	-	yellow	white	red	-	blue	yellow	white
Six	-	-	red	yellow	-	-	red	blue	-	-	white	red	-	-	yellow	white

W	B	R	Y
0	0	0	0
1	0	0	0
1	1	0	0
1	1	1	0
1	1	1	1
0	1	1	1
0	0	1	1

Specialization 3: Sam Beckett's "Quad" problem

Attempts to fix "Quad":

Y R B W

0 0 0 0
0 0 0 1
0 0 1 1
0 1 1 1
1 1 1 1
1 1 1 0
1 1 0 0
1 0 0 0
1 0 0 1
1 1 0 1
1 1 1 1
0 1 1 1
0 1 1 0
0 0 1 0
1 0 1 0
1 0 1 1
1 1 1 1
1 1 0 1
0 1 0 1
0 0 0 1

Specialization 3: Sam Beckett's "Quad" problem

Attempts to fix "Quad":

Y R B W

0 0 0 0
0 0 0 1
0 0 1 1
0 1 1 1
1 1 1 1
1 1 1 0
1 1 0 0
1 0 0 0
1 0 0 1
1 1 0 1
1 1 1 1
0 1 1 1
0 1 1 0
0 0 1 0
1 0 1 0
1 0 1 1
1 1 1 1 Repeated
1 1 0 1
0 1 0 1
0 0 0 1

Specialization 3: Sam Beckett's "Quad" problem

Attempts to fix "Quad":

Y R B W	Y R B W
0 0 0 0	0 0 0 0
0 0 0 1	0 0 0 1
0 0 1 1	0 0 1 1
0 1 1 1	0 0 1 0
1 1 1 1	0 1 1 0
1 1 1 0	0 1 0 0
1 1 0 0	1 1 0 0
1 0 0 0	1 0 0 0
1 0 0 1	1 0 1 0
1 1 0 1	1 1 1 0
1 1 1 1	1 1 1 1
0 1 1 1	0 1 1 1
0 1 1 0	0 1 0 1
0 0 1 0	1 1 0 1
1 0 1 0	1 0 0 1
1 0 1 1	1 0 1 1
1 1 1 1 Repeated	
1 1 0 1	
0 1 0 1	
0 0 0 1	

Specialization 3: Sam Beckett's "Quad" problem

Attempts to fix "Quad":

Y R B W		Y R B W
0 0 0 0		0 0 0 0
0 0 0 1		0 0 0 1
0 0 1 1		0 0 1 1
0 1 1 1		0 0 1 0
1 1 1 1		0 1 1 0
1 1 1 0		0 1 0 0
1 1 0 0		1 1 0 0
1 0 0 0		1 0 0 0
1 0 0 1		1 0 1 0
1 1 0 1		1 1 1 0
1 1 1 1		1 1 1 1
0 1 1 1		0 1 1 1
0 1 1 0		0 1 0 1
0 0 1 0		1 1 0 1
1 0 1 0		1 0 0 1
1 0 1 1		1 0 1 1
1 1 1 1	Repeated	Noncyclic
1 1 0 1		
0 1 0 1		
0 0 0 1		

Specialization 3: Sam Beckett's "Quad" problem

Attempts to fix "Quad":

Y R B W	Y R B W	Y R B W
0 0 0 0	0 0 0 0	0 0 0 0
0 0 0 1	0 0 0 1	0 0 0 1
0 0 1 1	0 0 1 1	0 0 1 1
0 1 1 1	0 0 1 0	0 0 1 0
1 1 1 1	0 1 1 0	0 1 1 0
1 1 1 0	0 1 0 0	0 1 0 0
1 1 0 0	1 1 0 0	1 1 0 0
1 0 0 0	1 0 0 0	1 0 0 0
1 0 0 1	1 0 1 0	1 0 1 0
1 1 0 1	1 1 1 0	1 1 1 0
1 1 1 1	1 1 1 1	1 1 1 1
0 1 1 1	0 1 1 1	0 1 1 1
0 1 1 0	0 1 0 1	0 1 0 1
0 0 1 0	1 1 0 1	1 1 0 1
1 0 1 0	1 0 0 1	1 0 0 1
1 0 1 1	1 0 1 1	1 0 0 0
1 1 1 1 Repeated	Noncyclic	
1 1 0 1		
0 1 0 1		
0 0 0 1		

Specialization 3: Sam Beckett's "Quad" problem

Attempts to fix "Quad":

Y R B W	Y R B W	Y R B W
0 0 0 0	0 0 0 0	0 0 0 0
0 0 0 1	0 0 0 1	0 0 0 1
0 0 1 1	0 0 1 1	0 0 1 1
0 1 1 1	0 0 1 0	0 0 1 0
1 1 1 1	0 1 1 0	0 1 1 0
1 1 1 0	0 1 0 0	0 1 0 0
1 1 0 0	1 1 0 0	1 1 0 0
1 0 0 0	1 0 0 0	1 0 0 0
1 0 0 1	1 0 1 0	1 0 1 0
1 1 0 1	1 1 1 0	1 1 1 0
1 1 1 1	1 1 1 1	1 1 1 1
0 1 1 1	0 1 1 1	0 1 1 1
0 1 1 0	0 1 0 1	0 1 0 1
0 0 1 0	1 1 0 1	1 1 0 1
1 0 1 0	1 0 0 1	1 0 0 1
1 0 1 1	1 0 1 1	1 0 0 0 ⟶ 1 0 1 1
1 1 1 1 Repeated	Noncyclic	
1 1 0 1		
0 1 0 1		
0 0 0 1		

Specialization 3: Sam Beckett's "Quad" problem

Attempts to fix "Quad":

Y R B W	Y R B W	Y R B W
0 0 0 0	0 0 0 0	0 0 0 0
0 0 0 1	0 0 0 1	0 0 0 1
0 0 1 1	0 0 1 1	0 0 1 1
0 1 1 1	0 0 1 0	0 0 1 0
1 1 1 1	0 1 1 0	0 1 1 0
1 1 1 0	0 1 0 0	0 1 0 0
1 1 0 0	1 1 0 0	1 1 0 0
1 0 0 0	1 0 0 0	1 0 0 0
1 0 0 1	1 0 1 0	1 0 1 0
1 1 0 1	1 1 1 0	1 1 1 0
1 1 1 1	1 1 1 1	1 1 1 1
0 1 1 1	0 1 1 1	0 1 1 1
0 1 1 0	0 1 0 1	0 1 0 1
0 0 1 0	1 1 0 1	1 1 0 1
1 0 1 0	1 0 0 1	1 0 0 1
1 0 1 1	1 0 1 1	1 0 0 0 $\longrightarrow$ 1 0 1 1
1 1 1 1 Repeated	Noncyclic	
1 1 0 1		
0 1 0 1		
0 0 0 1		

There exists no 4–bit Beckett Gray code!

Specialization 3: Sam Beckett's "Quad" problem

5–actor Quad	6–actor Quad	
00000	000000	110110
00001	000001	110111
00011	000011	010111
00010	000010	010011
00110	000110	011011
01110	001110	011001
11110	001111	001001
11100	001101	101001
11000	011101	111001
11001	011111	111000
10001	111111	111100
10101	111011	111101
00101	110011	110101
00100	110010	010101
01100	111010	000101
01101	111110	000111
11101	101110	100111
11111	101100	100011
11011	001100	100010
10011	011100	101010
10010	011110	101000
11010	010110	001000
01010	010010	011000
01000	011010	010000
01001	001010	010001
01011	001011	110001
01111	101011	100001
00111	101111	100000
10111	101101	110000
10110	100101	110100
10100	100100	010100
10000	100110	000100

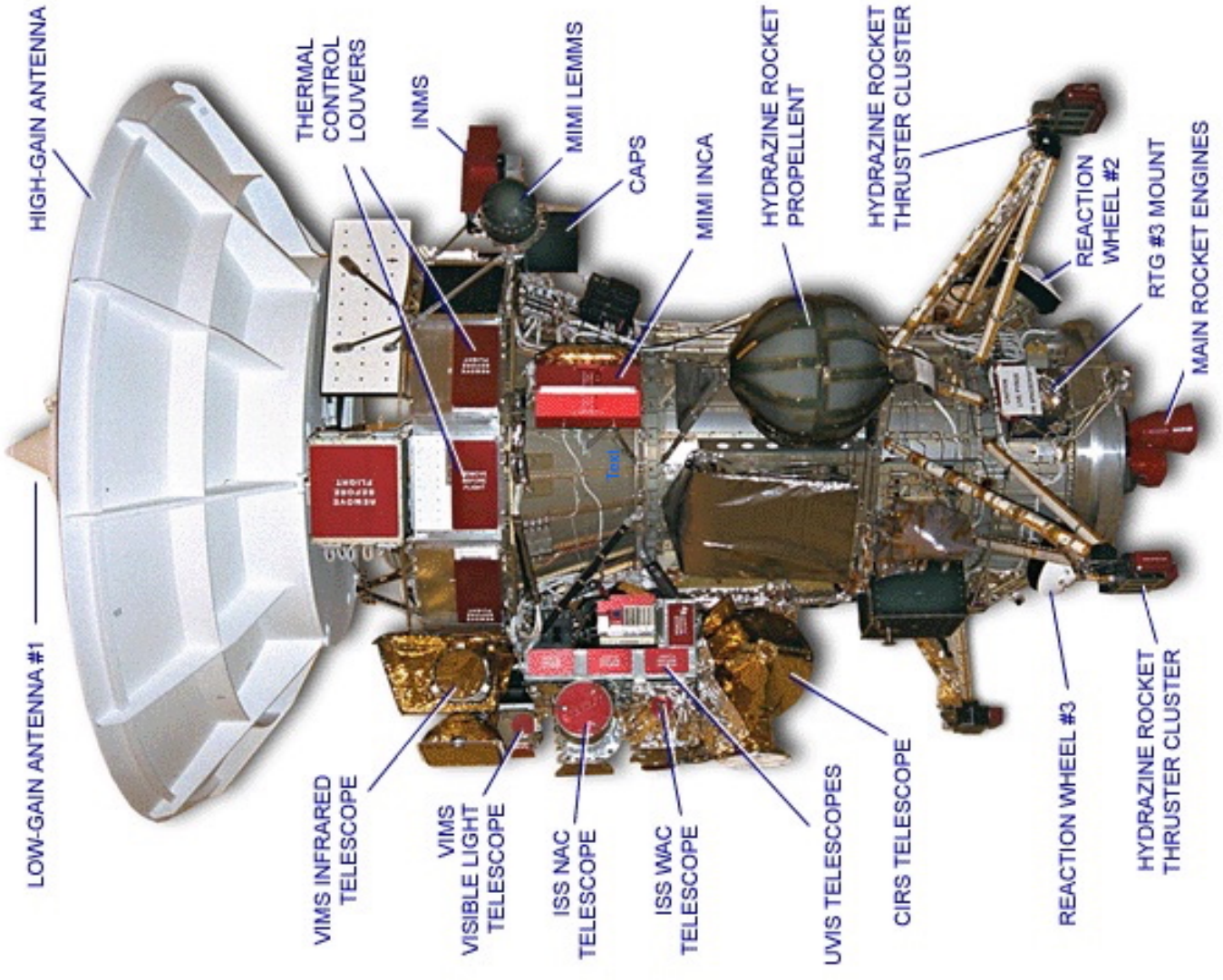
Current Status for n–bit Beckett Codes

1	Y	
2	Y	
3	N	
4	N	
5	Y	
6	Y	
7	Y	(Sawada & Wong, 2007)
8	Y	(B. Stevens, 2011)
>8	?	

Specialization 4: Minimum run-length problem

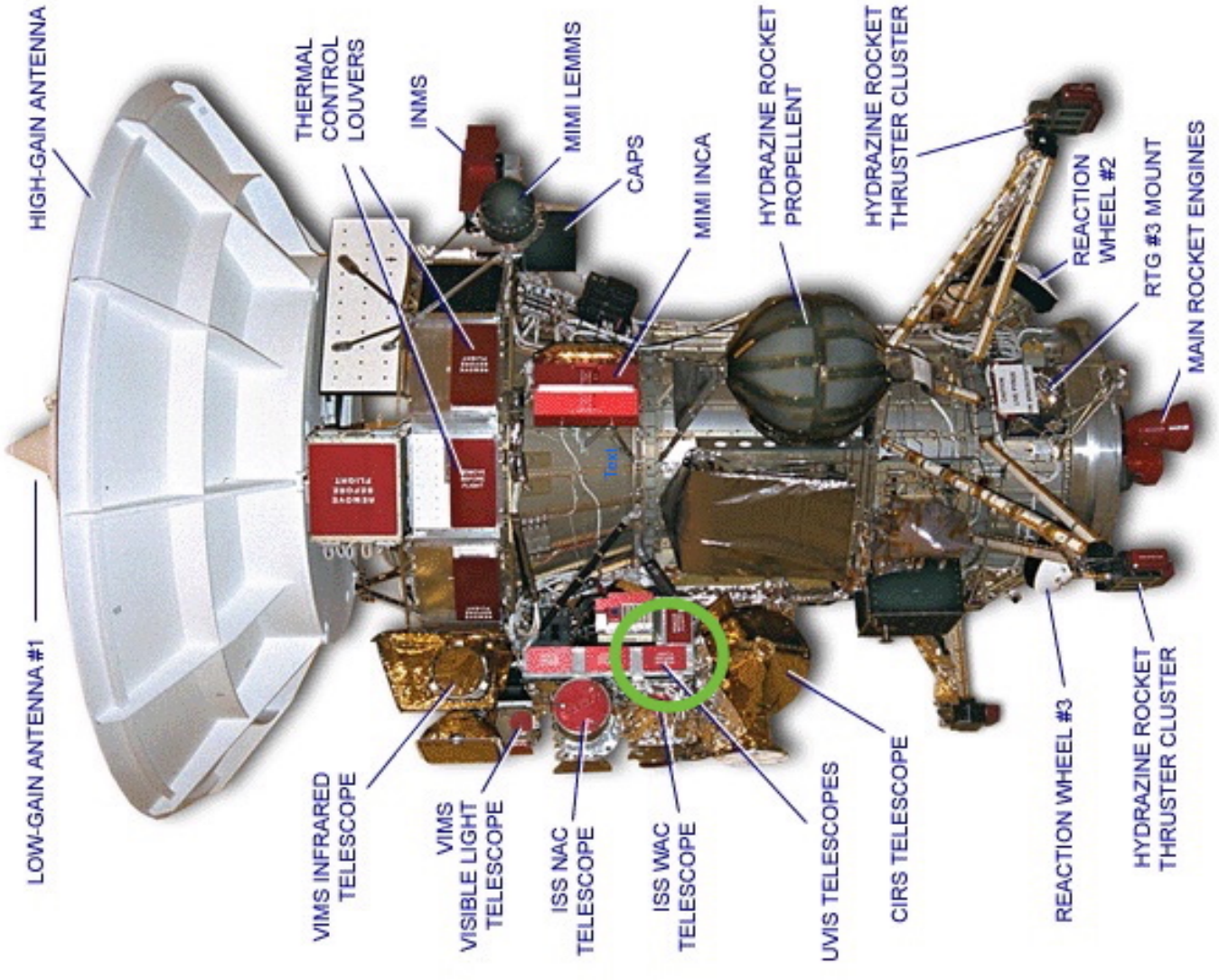
Specialization 4: Minimum run-length problem

CASSINI Satelite



Specialization 4: Minimum run-length problem

CASSINI Satelite



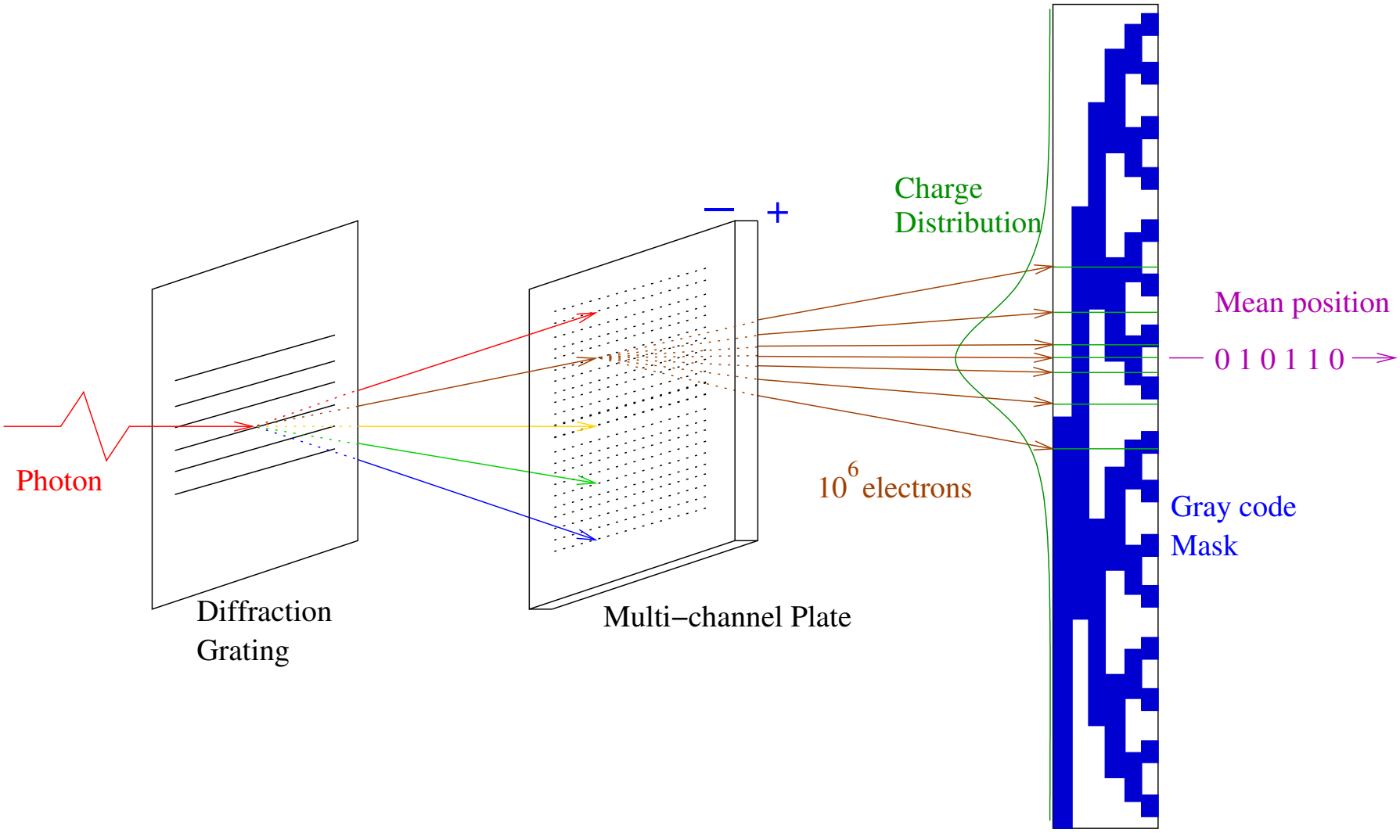
Specialization 4: Minimum run-length problem

CODACON Ultraviolet Spectrum Detector
(George Lawrence, LASP, Colorado, 1993)

Challenge: Extremely low flux
~ 2 photons per angstrom per cm^2 per second.

Specialization 4: Minimum run-length problem

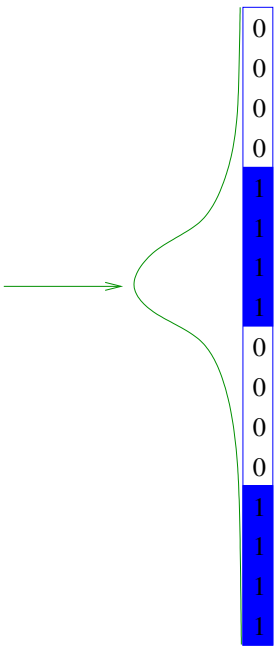
Detector Schematic



10 bit tracks, 1024 detectable UV frequencies

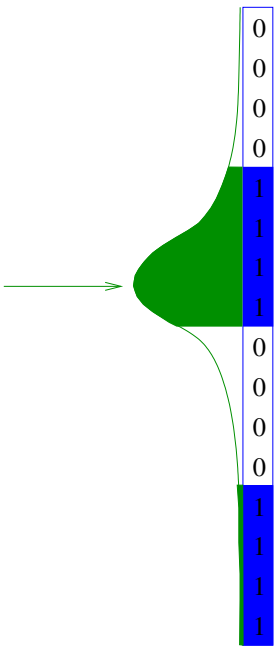
Specialization 4: Minimum run-length problem

Determining the mean position of 10^6 electrons
(10^5 electrons per bit track)



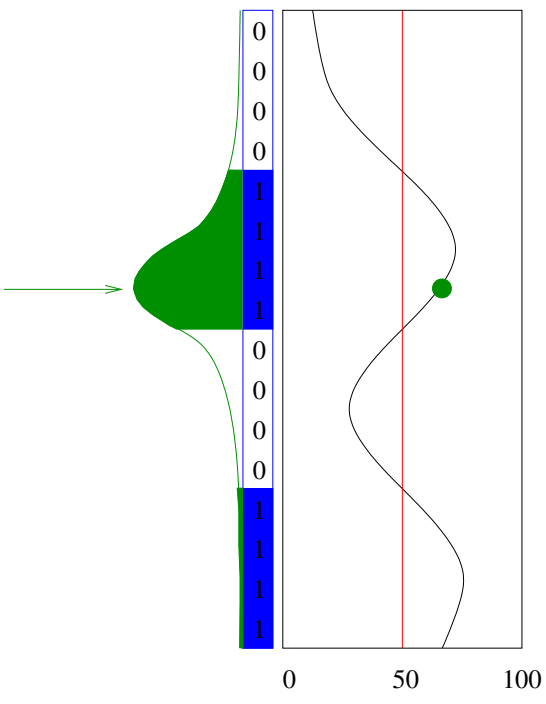
Specialization 4: Minimum run-length problem

Determining the mean position of 10^6 electrons
(10^5 electrons per bit track)



Specialization 4: Minimum run-length problem

Determining the mean position of 10^6 electrons
(10^5 electrons per bit track)



Specialization 4: Minimum run-length problem

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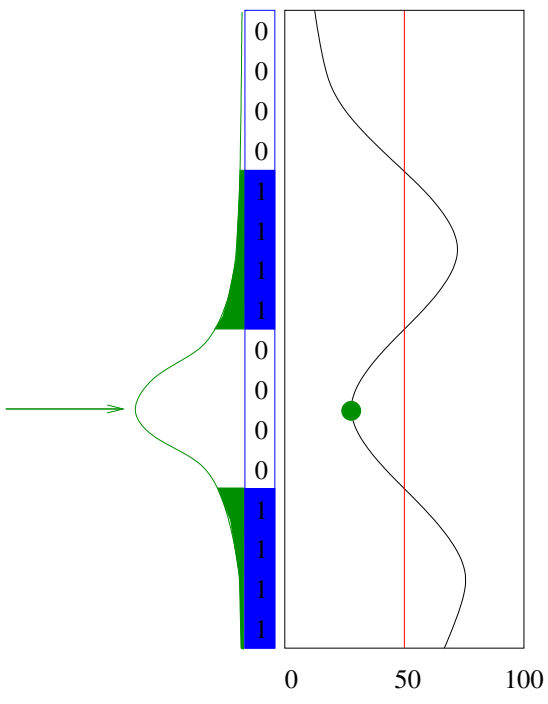

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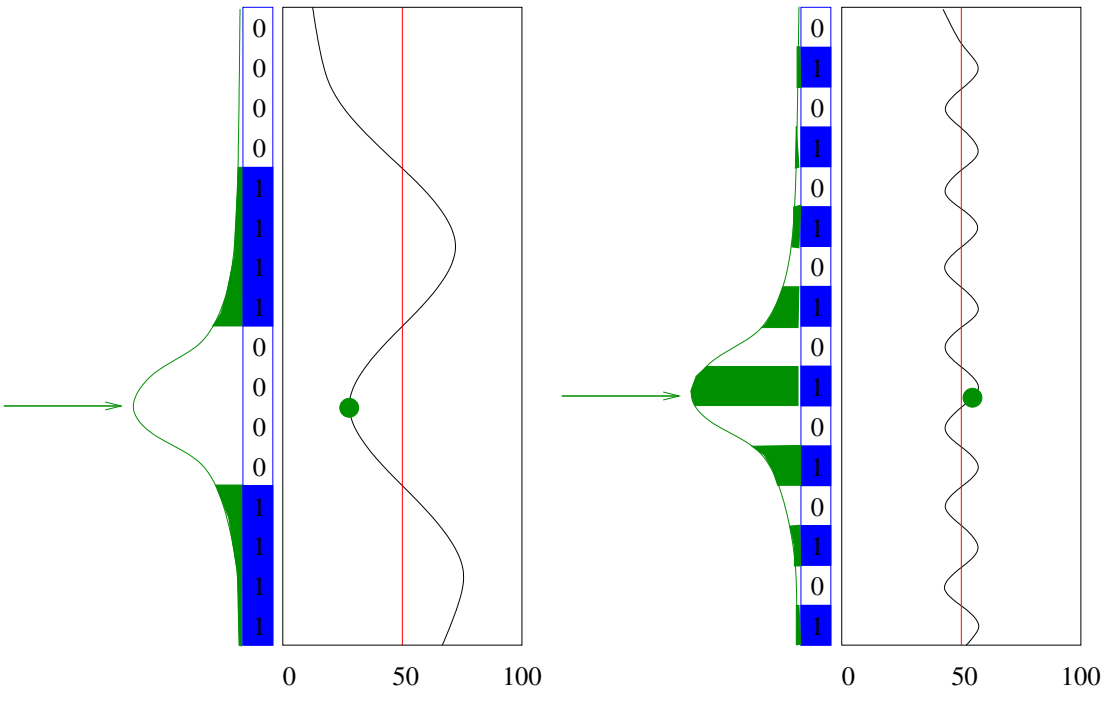

Specialization 4: Minimum run-length problem

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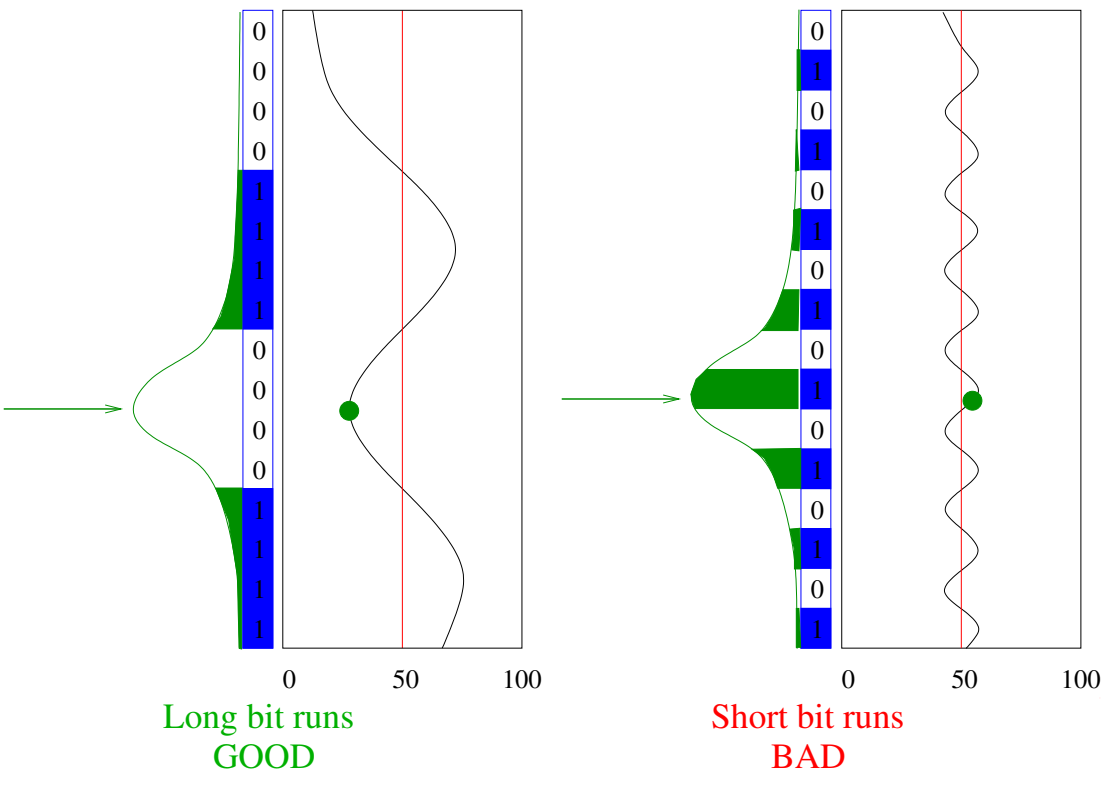

Specialization 4: Minimum run-length problem

Critical property:

If a bit changes ($0 \mapsto 1$ or $1 \mapsto 0$), then it must not change again for as long as possible.

Definition

Minimum run length:

$\mathrm{mrl}(C) =$ minimum length of a run of 0's or 1's (among all bit tracks in a Gray code C).

$\mathrm{mrl}(n) = \max\{\, \mathrm{mrl}(C) \mid C \text{ is an } n\text{-bit binary Gray code}\}$

Specialization 4: Minimum run-length problem

Proposition

If $n \geq 3$, *then* $\mathrm{mrl}(n) \leq n - 1$.

Proof.

$$\mathrm{mrl}(C) \leq \text{ average run length}(C) = n2^n/2^n = n.$$

If $\mathrm{mrl}(C) = n$, then $|C| \leq 2n < 2^n$.

$$C = \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Specialization 4: Minimum run-length problem

Theorem (Goddyn, Gvozdjak 2003)

$\mathrm{mrl}(n) \geq n - \lfloor 2.0001 \lg n \rfloor$

Proof.

- Basis: By computer,

n	1	2	3	4	5	6	7
$\mathrm{mrl}(n)$	1	2	2	2	4	4	5

$$C_5 = [1425143524152435142514352415243 5]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ & . & . & . & \end{bmatrix}$$

Specialization 4: Minimum run-length problem

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Proof.

- Basis: By computer,

n	1	2	3	4	5	6	7
$\mathrm{mrl}(n)$	1	2	2	2	4	4	5

- Recurrence relation on $Q_{a+b} = Q_a \times Q_b$:

  If $(a-1)(2^a - 2a - 6) \leq 2b$, then
  $\mathrm{mrl}(a+b) \geq 2 \min\{a-1, \mathrm{mrl}(b)\}$.

Specialization 4: Minimum run-length problem

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  [Merge a "stream" of walks in Q_a with an optimal Gray code in Q_b by careful interleaving.]

□

Specialization 4: Minimum run-length problem

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  If $(a-1)(2^a - 2a - 6) \leq 2b$, then
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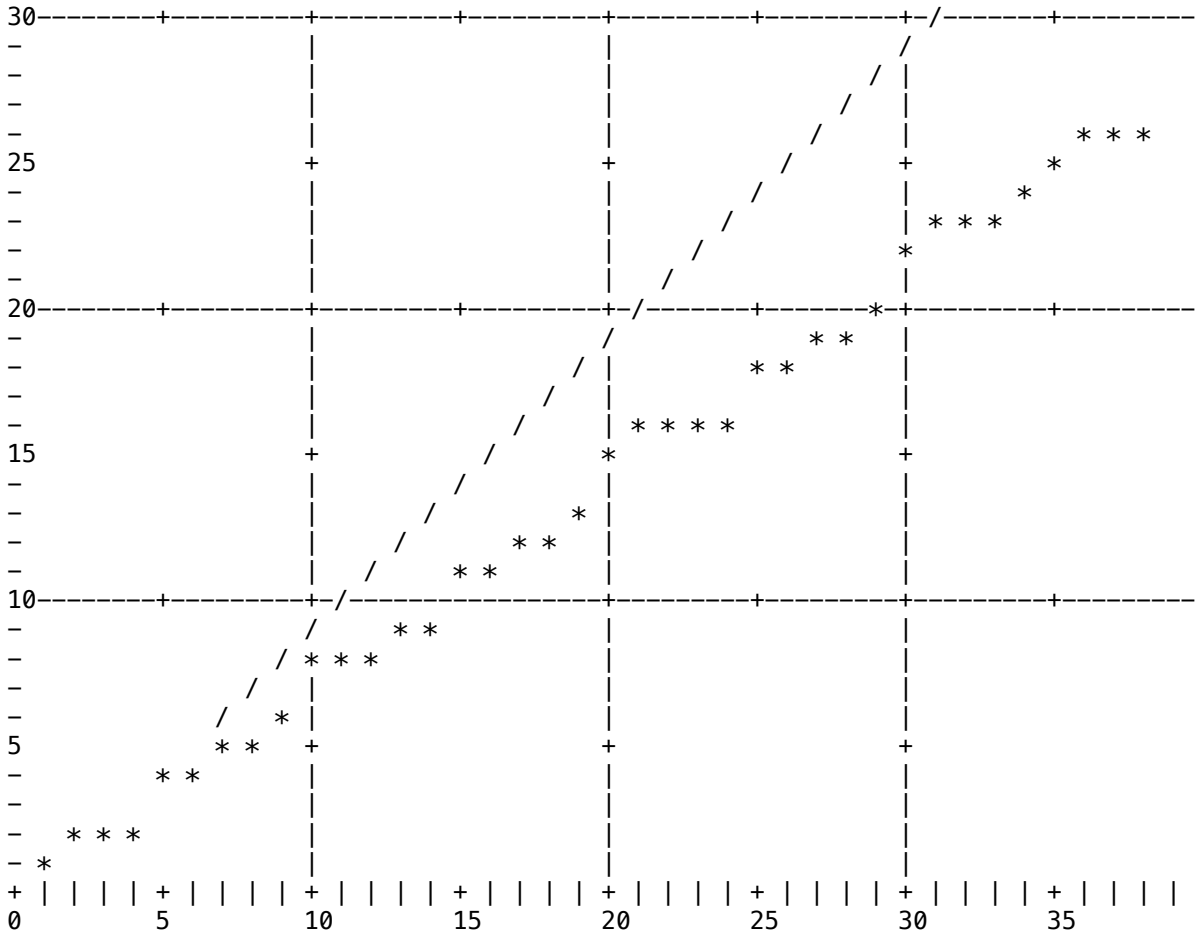
- Solve the recurrence.

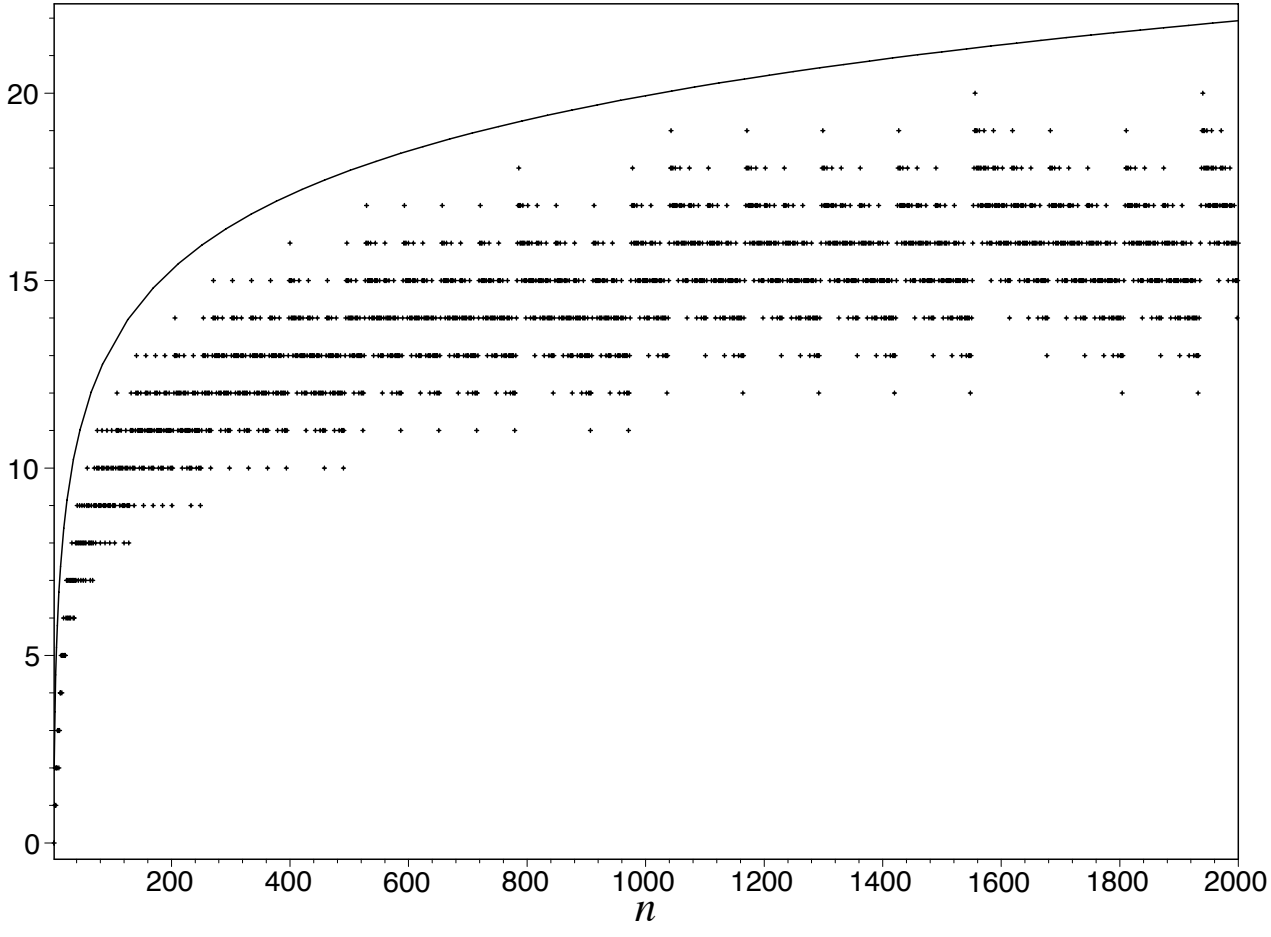
□

Specialization 4: Minimum run-length problem



mrl(n) versus n

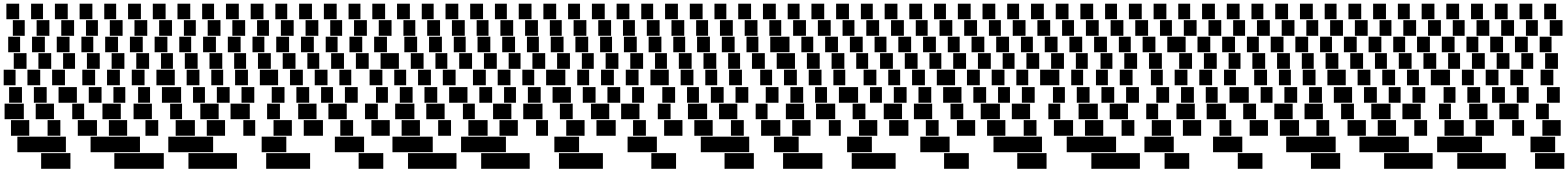
Specialization 4: Minimum run-length problem



Upper bounds on $n - \mathrm{mrl}(n)$ versus n

Specialization 4: Minimum run-length problem

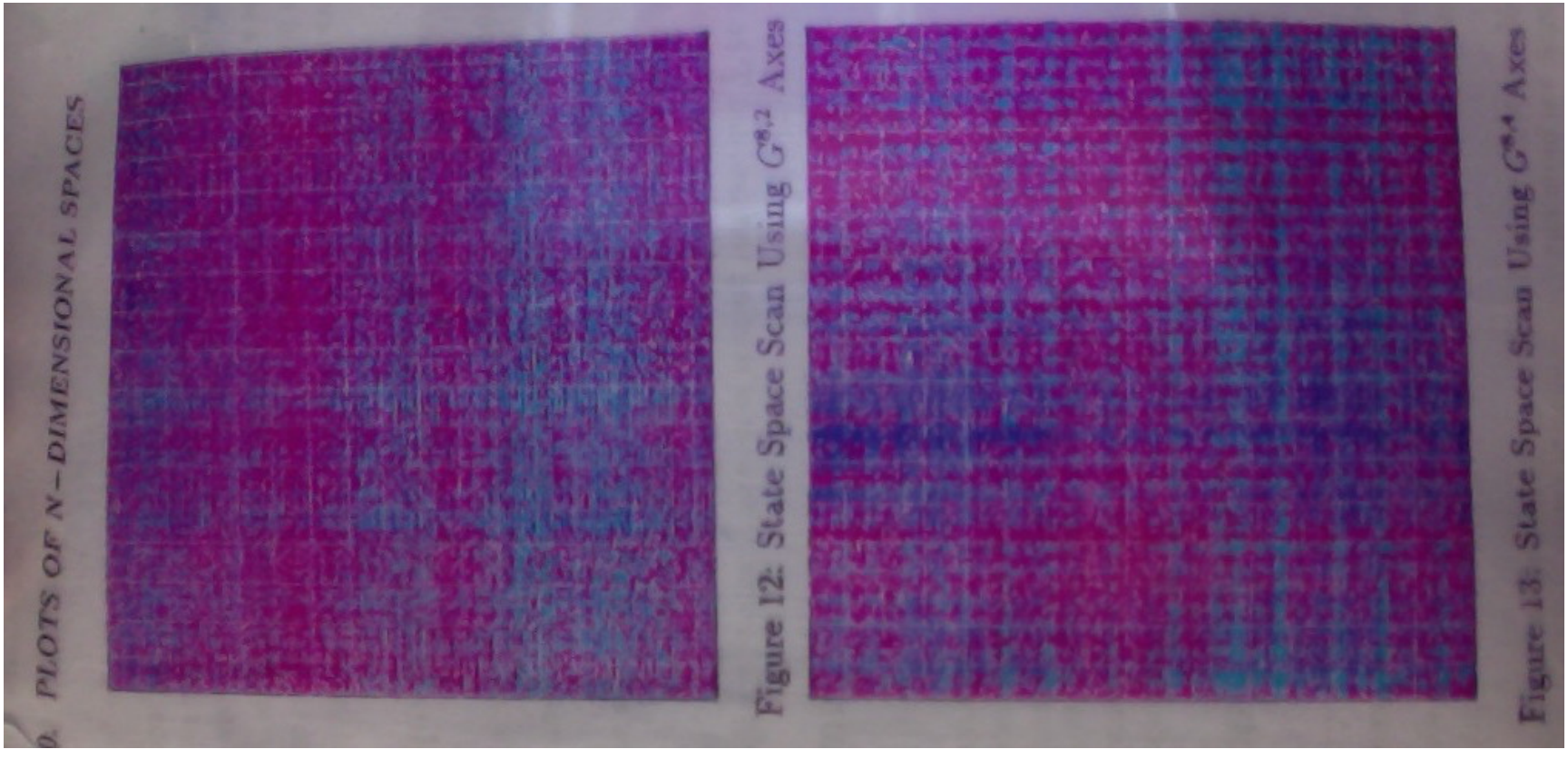
10-bit Gray code with Minimum Run Length 8



CODACON Template: 250% improvement on resolution.

MRL Application: Neural Net Visualization

Local distance preserving



$$C = v_0 v_1 \dots v_{2^n-1} \qquad \mathrm{mrl}(C) = m$$

$$|i-j| \leq m \implies d(v_i, v_j) = |i-j|$$

MRL Open Problem:

Find the *longest path* in Q_n having minimum run length $n-1$.

n	1	2	3	4	5	6	7	8
$L(n)$	1	4	6	13	32	49	113	≥ 223

- Does $\frac{L(n)}{2^n} \longrightarrow 1$?
- Explain mysterious symmetries in optimal solutions.

All 6-bit paths (and cycles) of MRL 5 and of length 49:

Type A

```
  !   !! !!!  ! !!!! !! !!!! !  !!! !!   !      11
 !!   !! !!!  ! !!!! !! !!!! !  !!! !!   !!     22
!!!   !! !!!  ! !!!! !! !!!! !  !!! !!   !!!    33

 !   !! !!!  ! !!!! !! !!!! !  !!! !!   !!     12
!!   !! !!!  ! !!!! !! !!!! !  !!! !!   !      21

 !   !! !!!  ! !!!! !! !!!! !  !!! !!   !!!    13
!!!  !! !!!  ! !!!! !! !!!! !  !!! !!   !      31

!!   !! !!!  ! !!!! !! !!!! !  !!! !!   !!!    23
!!!  !! !!!  ! !!!! !! !!!! !  !!! !!   !!     32
```

Type B

```
!!    ! !!!! !! !!!! !! !!!! !! !!!! !    !!    11
!!!   !  !!! !! !!!! !! !!!! !! !!!  !   !!!    22
 !    ! !!!! !! !!!! !! !!!! !! !!!! !    !     33
 !!   !  !!! !! !!!! !! !!!! !! !!!  !   !!    44
!! !  ! !!!! !! !!!! !! !!!! !! !!!! !  ! !!   55
```

Conclusions/Questions

- Many more applications require specialized Gray codes.
  - statistical design
  - algorithm design
  - hardware design
  - neural networks
  - communication encoding
  - efficient search and generation in big data
- Computer searches:
  - Explain/exploit apparent symmetries in optimal Gray codes.
  - Find general constructions (linear time count, Beckett codes).
- Find optimal minimal change orders for other mathematical objects: permutations, partitions, colourings, rooted trees,. . .